Standard Model/Quantun Field Theory

Solution Set 4

Due: 18 October 2019

Suggested reading: QFT Notes Ch 22; Textbook: Secs. 69-73

10. In the lecture notes we listed the eight Gell-Mann lambda matrices λ_a in Eq (22.19). Then the generators of $SU(3)$ in the (defining) 3 dimensional representation are given by $T_a = \lambda_a/2$.

a) Confirm that these T_a satisfy $\mathrm{Tr}T_aT_b = \delta_{ab}/2$.

Solution: We need to show $\mathrm{Tr}\lambda_a\lambda_b = 2\delta_{ab}$. It is almost immediate that for $a = 1, \ldots, 7$, λ_a^2 is a diagonal matrix with two diagonal entries of 1 and one of 0, so its trace is 2. λ_8^2 =diag$(1, 1, 4)/3$ so its trace is also 2 Thus it remains to show that $\mathrm{Tr}\lambda_a\lambda_b = 0$ for $a \neq b$. This is automatic if one matrix is symmetric $(\lambda_1, \lambda_3, \lambda_4, \lambda_6.\lambda_8)$ and the other is antisymmetric $\lambda_2, \lambda_5, \lambda_7)$. λ_3 or λ_8 multiplied by any of the other 6 has no diagonal entries and hence is traceless. Further $\lambda_3\lambda_8$ =diag$(1, -1, 0)/\sqrt{3}$ is obviously traceless. By inspection $\lambda_1\lambda_4, \lambda_1\lambda_6, \lambda_4\lambda_6, \lambda_2\lambda_5, \lambda_2\lambda_7, \lambda_5\lambda_7$ all have zero diagonal entries and hence zero trace.

b) Use this explicit representation to evaluate the structure constants f_{abc} for $SU(3)$. Remember that the f_{abc} are antisymmetric under the exchange of any pair of indices.

Solution: It is immediate that if a, b, c are all from the set $1, 2, 3$, $f_{abc} = \epsilon_{abc}$. Furthermore if any pair are from the set $1, 2, 3$ and the third is not, then $f_{abc} = 0$. Now λ_8 commutes with $\lambda_{1,2,3}$, so $f_{18a} = f_{28a} = f_{38a} = 0$. Now we calculate

$$
\begin{aligned}
[\lambda_8, \lambda_4] &= \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} - h.c. = i\sqrt{3}\lambda_5 \\
[\lambda_8, \lambda_5] &= \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -2i & 0 & 0 \end{pmatrix} - h.c. = -i\sqrt{3}\lambda_4
\end{aligned}
$$

from which $f_{84a} = \sqrt{3}\delta_{a5}/2$ and $f_{85a} = -\sqrt{3}\delta_{4a}/2$. Similarly $f_{86a} = \sqrt{3}\delta_{a7}/2$ and $f_{87a} = -\sqrt{3}\delta_{a6}/2$. Next

$$
\begin{aligned}
[\lambda_3, \lambda_4] &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - h.c. = i\lambda_5 \\
[\lambda_3, \lambda_5] &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - h.c. = -i\lambda_4
\end{aligned}
$$

from which $f_{34a} = \delta_{a5}/2$ and $f_{35a} = -\delta_{a4}/2$. Similarly $f_{36a} = -\delta_{a7}/2$ and $f_{37a} = \delta_{a6}/2$. Next

$$
\begin{aligned}
[\lambda_1, \lambda_4] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - h.c. = i\lambda_7 \\
[\lambda_1, \lambda_5] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & 0 & 0 \end{pmatrix} - h.c. = -i\lambda_6
\end{aligned}
$$

from which $f_{14a} = \delta_{a7}/2$ and $f_{15a} = -\delta_{a6}/2$. Similarly $f_{16a} = \delta_{a5}/2$ and $f_{17a} = -\delta a4/2$. Next

$$
\begin{aligned}
[\lambda_2, \lambda_4] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix} - h.c. = i\lambda_6 \\
[\lambda_2, \lambda_5] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - h.c. \ = i\lambda_7
\end{aligned}
$$

from which $f_{24a} = \delta_{a6}/2$ and $f_{25a} = \delta_{a7}/2$. In summary the nonzero structure constants are:

$$f_{123} = 1, \quad f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}, \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2} \tag{1}$$

11. Consider a general Lie group with generators T^a, For a general representtion R assume they are orthonormal $\mathrm{Tr} T^a_R T^b_R = T(R)\delta_{ab}$. The Casimir operator is defined by $C(R) = \sum_a T^a_R T^a_R$ or with summation convention $C(R) = T^a_R T^a_R$

a) Show that $[C(R), T^b_R] = 0$. Thus it has the same value on all states in an irreducible representation. E.g. for the rotation group it is just $\boldsymbol{J}^2$ and has the value $j(j+1)$ in a representation of spin j.

Solution: $[T^a T^a, T^b] = T^a i f_{abc} T^c + i f_{abc} T^c T^a = T^a i f_{abc} T^c - i f_{cba} T^c T^a = 0$ by the antisymmetry of f and renaming dummy indices.

b) Let $D(R)$ be the dimension of the representation R, and denote the Adjoint representation by $R = A$. Prove that $T(R)D(A) - C(R)D(R)$.

Solution: By definition $\mathrm{Tr} T^a(R) T^b(R) = \delta_{ab} T(R)$. Set $b = a$ and summ all a: $\mathrm{Tr} C(R) = D(A)T(R)$. But $\mathrm{Tr} C(R) = C(R)D(R)$. The desired result follows.

c) Remembering that the nonabelian field strength transforms in the adjoint representation prove the Bianchi identity:

$$D_\mu F_{\rho\sigma} + D_\rho F_{\sigma\mu} + D_\sigma F_{\mu\rho} = 0$$

Solution: The easiest way to prove the Bianchi identity is to recall that if D_μ is a matrix representation of a covariant derivative, then the field strength as a matrix is simply $F_{\mu\nu} =$

$[D_\mu, D_\mu]$. Furthermore the covariant derivative is simply $[D_\rho, F_{\mu\nu}]$. Using these facts the Bianchi identity reads

$$[D_\rho, [D_\mu, D_\nu]] + [D_\nu, [D_\rho, D_\mu]] + [D_\mu, [D_\nu, D_\rho]] - 0 \tag{2}$$

This is nothing but the Jacobi identity satisfied by commutators, so the Bianchi identity is established.

12. Using dimensional regularization, calculate the one loop self energy diagram of a Dirac fermion in a general representation of the gauge group coupled to a general (nonabelian) gauge field in the ξ gauge, i.e. the gauge field propagator is

$$-i\delta_{ab}\frac{\eta_{\mu\nu} + (\xi - 1)k_\mu k_\nu/k^2}{k^2 - i\epsilon}. \tag{3}$$

Assume the fermion momentum p is off-shell, i.e. $p^2 \neq -m^2$ so the integral will be finite in the infrared. Calculate the residue of the pole at $D = 4$ and comment on the simplification that occurs for $\xi = 0$ (Landau Gauge).

Solution: Let T_a be the representatives of the gauge group. Then the Loop integral to be done is

$$\begin{aligned}
&(-i)^2 \int \frac{d^D k}{(2\pi)^D} \frac{ig\gamma^\mu T_a(m - \gamma\cdot(p-k))ig\gamma^\nu T_b \delta_{ab}(\eta_{\mu\nu}k^2 - (1-\xi)k_\mu k_\nu)}{(k^2 - i\epsilon)^2(m^2 + (p-k)^2 - i\epsilon)} \\
= \ & 2\frac{g^2 T^{a2}}{(2\pi)^D}\int d^D k \int_0^1 dx (1-x) \frac{\gamma^\mu(m - \gamma\cdot(p-k))\gamma^\nu(\eta_{\mu\nu}k^2 - (1-\xi)k_\mu k_\nu)}{[(k-xp)^2 + xm^2 + x(1-x)p^2) - i\epsilon)]^3}
\end{aligned} \tag{4}$$

After the shift $k \to k + xp$ the numerator becomes

$$\begin{aligned}
&\gamma^\mu(m - \gamma\cdot(p(1-x) - k))\gamma^\nu(\eta_{\mu\nu}(k+xp)^2 - (1-\xi)(k+xp)_\mu(k+xp)_\nu) \\
&= [-Dm - (D-2)\gamma\cdot(p(1-x)-k)](k+xp)^2 - (1-\xi)\gamma\cdot(k+xp)(m - \gamma\cdot((1-x)p-k))\gamma\cdot(k+xp) \\
&\to (k^2 + x^2p^2)[-Dm - (D-2)\gamma\cdot p(1-x)] + 2x(D-2)k\cdot p\gamma\cdot k \\
&-(1-\xi)\gamma\cdot(xp)(m - \gamma\cdot((1-x)p))\gamma\cdot(xp) - (1-\xi)\gamma\cdot k(m - \gamma\cdot((1-x)p))\gamma\cdot k \\
&-(1-\xi)\gamma\cdot k(\gamma\cdot k)\gamma\cdot(xp) - (1-\xi)\gamma\cdot xp(\gamma\cdot k)\gamma\cdot(k) \\
&\to (k^2 + x^2p^2)[-Dm - (D-2)\gamma\cdot p(1-x)] + 2x(D-2)k^2 p\cdot\gamma/D \\
&-(1-\xi)\gamma\cdot(xp)(m - \gamma\cdot((1-x)p))\gamma\cdot(xp) - (1-\xi)k^2(-Dm - (D-2)\gamma\cdot((1-x)p))/D \\
&+2xk^2(1-\xi)\gamma\cdot p \\
&= k^2\big[-Dm - (D-2)\gamma\cdot p(1-x) + 2x(D-2)p\cdot\gamma/D \\
&-(1-\xi)(-Dm - (D-2)\gamma\cdot((1-x)p))/D + 2x(1-\xi)\gamma\cdot p\big] \\
&+x^2p^2[-Dm - (D-2)\gamma\cdot p(1-x) - (1-\xi)(-m + (1-x)\gamma\cdot p)]
\end{aligned}$$

In the second step we dropped odd powers of k and in the third averaged over directions of k. Write the numerator as $k^2(Am + B\gamma\cdot p) + x^2p^2(Cm + E\gamma\cdot p)$.

$$\begin{aligned}
A &= 1-\xi-D \to 1-D \\
B &= (-D+2)(1-x) + 2x\frac{D-2}{D} + (1-\xi)(1-x)\frac{D-2}{D} + 2x(1-\xi) \\
&\to (-D+2)(1-x) + (1+x)\frac{D-2}{D} + 2x \\
C &= 1-\xi-D \to 1-D \\
E &= (-D+1+\xi)(1-x) \to (1-D)(1-x)
\end{aligned}$$

where the arrows give Landau gauge. Then after the Wick rotation the diagram is

$$2i\frac{g^2T^{a2}}{(2\pi)^D}\frac{2\pi^{D/2}}{\Gamma(D/2)}\int_0^1 dx(1-x)\int_0^\infty k^{D-1}dk\frac{k^2(Am+B\gamma\cdot p)+p^2(Cm+E\gamma\cdot p)}{[k^2+xm^2+x(1-x)p^2)]^3} \tag{5}$$

To do the k integral we need

$$\begin{aligned}
2\int_0^\infty dkk^{d-1}\frac{k^n}{(k^2+G)^m} &= G^{1+(D-2+n)/2-m}\int_0^\infty duu^{(D-2+n)/2}(u+1)^{-m} \\
\int_0^\infty duu^\alpha(1+u)^{-m} &= \int_1^\infty duu^{-m}(u-1)^\alpha = \int_0^1 dvv^{m-2-\alpha}(1-v)^\alpha \\
&= \frac{\Gamma(m-1-\alpha)\Gamma(\alpha+1)}{\Gamma(m)}
\end{aligned} \tag{6}$$

for $m=3$ and $n=2,0$, which are $G^{D/2-2}\Gamma(2-D/2)\Gamma(1+D/2)/\Gamma(3)$ and $G^{D/2-3}\Gamma(3-D/2)\Gamma(D/2)/\Gamma(3)$ respectively. Putting $G = xm^2 + x(1-x)p^2$ gives

$$\begin{aligned}
&i\frac{2g^2T^{a2}}{2^D\pi^{D/2}}\Gamma(3-D/2)\int_0^1 dx(1-x)(xm^2+x(1-x)p^2)^{D/2-2} \\
&\qquad\left[\frac{D}{4-D}(Am+B\gamma\cdot p) + \frac{xp^2}{m^2+(1-x)p^2}(Cm+E\gamma\cdot p)\right]
\end{aligned} \tag{7}$$

for the value of the diagram. There is a pole at $D=4$ whose residue is

$$i\frac{g^2T^{a2}}{8\pi^2}\int_0^1 dx(1-x)\left[\frac{4}{4-D}(A(D=4)m + B(D=4)\gamma\cdot p)\right] \tag{8}$$

Now $A(D=4) = -\xi-3$ and $B(D=4) = -3(1-3x)/2 - \xi(3x+1)/2$ so that

$$\begin{aligned}
\int dx(1-x)A(D=4) &= -\frac{3+\xi}{2} \\
\int dx(1-x)B(D=4) &= -\frac{\xi}{2}\int dx(1+2x-3x^2) = -\frac{\xi}{2}
\end{aligned} \tag{9}$$

Then the residue becomes

$$-i\frac{g^2T^{a2}}{8\pi^2}\frac{2}{4-D}\left[3m + \xi(m+\gamma\cdot p)\right] \tag{10}$$

From whixh we see that the residue is independent of momentum in Landau gauge ($\xi = 0$). This means that Z_2 is finite in Landau gauge. This was important in our class discussion of asymptotic freedom, in dispensing with two of the QED like diagrams: the fermi self energy and the QED-like vertex diagram.