Standard Model/Quantum Field Theory

Problem Set 5

Due: 1 November 2019

Suggested reading: QFT Notes Ch 22-23; Textbook: Secs. 74,87-88.

13. Consider a pure gauge theory with no extra "matter fields" for example QCD with no quarks. Call the gauge bosons gluons. Draw all the tree diagrams contributing to elastic gluon-gluon scattering and evaluate them in terms of the polarization vectors of the gluons and their color indices. Check gauge invariance by replacing one of the polarization vectors, say e_1 by p_1 and confirming that this gives 0.

Solution There are three gluon exchange graphs and a graph with a four point vertex. We construct a particular gluon exchange graph. Let a, b, c, d be the color of each gluon with momenta and polarizations p_k, ϵ_k with $k = 1, 2, 3, 4$ respectively. The graph where $1, 2$ enter at one vertex and $3, 4$ enter at the other vertex is

$$\begin{aligned}&(-g)^2 f_{abe} f_{cde}[2p_1 \cdot \epsilon_2 \epsilon_1^\sigma + (p_2 - p_1)^\sigma \epsilon_1 \cdot \epsilon_2 - 2p_2 \cdot \epsilon_1 \epsilon_2^\sigma]\\&[2p_3 \cdot \epsilon_4 \epsilon_{3\sigma} + (p_4 - p_3)_\sigma \epsilon_3 \cdot \epsilon_4 - 2p_4 \cdot \epsilon_3 \epsilon_{4\sigma}] \frac{-i}{(p_1 + p_2)^2}\end{aligned} \tag{1}$$

the other two gluon exchange graphs are obtained from this by substitution $(a, 1) \to (b, 2), (b, 2) \to (c, 3), (c, 3) \to (d, 4), (d, 4) \to (a, 1)$ for the first and $(b, 2) \to (c, 3), (c, 3) \to (b, 2)$. for the second. The quartic vertex diagram is given by

$$\begin{aligned}&-ig^2[f_{eab} f_{ecd}(\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3) + f_{eac} f_{ebd}(\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3)\\&+f_{ead} f_{ebc}(\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4)]\end{aligned} \tag{2}$$

A simple check is to replace say ϵ_1 by p_1 and see that it gives 0: In the exchange graph evaluation above the first two terms inside square brackets on the first line combine to be proportional to $(p_1 + p_2)^\sigma = -(p_3 + p_4)^\sigma$ which dotted into the second line give zero. The remaining term on the top line has a factor $2p_1 \cdot p_2 = (p_1 + p_2)^2$ which cancels the propagator leaving a remainder that is linear in momentum. The other two exchange graphs (not shown) also leave behind a remainder. All the remainders then combine to cancel the quartic vertex diagram with the help of the Jacobi identity.

14. In diagonalizing the fermion mass matrices in the standard model we needed to know that any complex square matrix M can be brought to diagonal form with nonnegative diagonal entries by a transformation $UMV^\dagger$ with U and V a pair of unitary matrices. Prove this.

Solution $MM^\dagger$ and $M^\dagger M$ are hermitian matrices which can be diagonalized with (different unitary matrices), $MM^\dagger = U^\dagger D_1 U$ and $M^\dagger M = V^\dagger D_2 V$, where D_1, D_2 are diagonal matrices with nonnegative entries. But $MM^\dagger$ and $M^\dagger M$ have identical eigenvalue spectra: if $MM^\dagger|\psi\rangle = \lambda|\psi\rangle$ then for $\lambda \neq 0$, $M^\dagger M(M^\dagger|\psi\rangle) = \lambda(M^\dagger|\psi\rangle)$, and similarly with M and $M^\dagger$ interchanged. If all the nonzero eigenvalues are the same they have the same number of zero eigenvectors as well. Thus D_1

and D_2 differ inly by a permutation of diagonal entries. So wolog we can take $D_1 = D_2$. Then it follows that

$$UMM^\dagger U^\dagger = VM^\dagger MV^\dagger \tag{3}$$

$$UMV^\dagger VM^\dagger U^\dagger = VM^\dagger U^\dagger UMV^\dagger \tag{4}$$

The last line shows that $UMV^\dagger$ commutes with its hermitian conjugate $VM^\dagger U^\dagger$. Thus $UMV^\dagger = W^\dagger DW$. where D can be taken to have positive diagonal entries. Finally we get $M = U^\dagger W^\dagger DWV = (WU)^\dagger DWV$ which was to be proved.

15. S, Problem 87.2

Solution:

a) We use the tree level relations $M_W^2 = v^2 g_2^2/2$, $M_Z^2 = M_W^2/\cos^2\theta_W$, $g_1 = g_2\tan\theta_W$, $e = g_2\sin\theta_W$. From the last relation we get $g_2 = e/\sin\theta_W = \sqrt{4\pi/127.9/0.231} \approx 0.652$. $g_1 = g_2\tan\theta_W = 0.652 * 0.598 = 0.357$. Finally $v = \sqrt{2}M_W/g_2 = 174\text{GeV}$

b) $G_F = g_2^2/(4\sqrt{2}M_W^2) = (0.652/80.4)^2/4/\sqrt{2} = 1.16 * 10^{-5}\text{Gev}^{-2}$.

c) $M_W^2 = v^2 g_2^2/2$ and $e = g_2\sin\theta_W$, so $G_F = 1/(2\sqrt{2}v^2)$. Note that Srednicki's v^2 is a factor of 2 times mine.

16. Consider the standard model Lagrangian using 't Hooft ξ gauge in the presence of the Higgs mechanism as specified in Eq, (22.81) of the Lecture notes.

a) By examining the terms quadratic in the scalar fields, work out the masses for the fluctuation fields $\hat{\phi} = \phi - v$. Assume $v^T = (v, 0)$, with v real, as we did in class. Show that their squares are proportional to ξ such that the masses coincide with the associated gauge boson masses when $\xi = 1$. Remember that the components of $\hat{\phi}$ are complex.

Solution: The quadratic terms of the first of (22.81) are, writing $\hat{\phi}^\dagger v - v^\dagger\hat{\phi} = v(\phi_1^* - \phi_1)$,

$$\frac{\xi g_1^2}{8}(-2i\text{Im}\phi_1)^2 = -\frac{\xi g_1^2}{2}(\text{Im}\phi_1)^2 \tag{5}$$

There is also a quadratic term in $\text{Im}\phi_1$ coming from the $a = 3$ part of the second equation of (22.81) which adds g_2^2 to g_1^2. To infer the mass we compare this to the quadratic derivative terms of the Lagrangian

$$-\partial\phi_1^\dagger \cdot \partial\phi_1 = -(\partial(\text{Re}\phi_1)^2 - -(\partial\text{Im}\phi_1)^2 \tag{6}$$

from which we see tht the mass squared of ϕ_1 is

$$\frac{\xi(g_1^2 + g_2^2)v^2}{2} = \frac{\xi g_2^2 v^2}{2\cos^2\theta_W} = \xi M_Z^2.$$

The quadratic terms of the second equation for $a = 1, 2$ are, writing $\hat{\phi}^\dagger(\tau_1/2)v - v^\dagger\hat{\phi}(\tau_1/2) == (v/2)(\phi_2^* - \phi_2)$ and $\hat{\phi}^\dagger(\tau_2/2)v - v^\dagger\hat{\phi}(\tau_2/2) == -i(v/2)(\phi_2^* + \phi_2)$,

$$-\frac{\xi v^2 g_2^2}{2}\left[(\text{Im}\phi_2)^2 + (\text{Re}\phi_2)^2\right] = -\frac{\xi v^2 g_2^2}{2}\phi_2^\dagger\phi_2 \tag{7}$$

Comparing to the derivative term $-\partial\phi_2^* \cdot \partial\phi_2$ we learn that the mass squared of the complex field ϕ_2 is

$$\frac{\xi v^2 g_2^2}{2} = \xi M_W^2. \tag{8}$$

b) By considering the transformation of the gauge fixing function under infinitesimal gauge transformations work out the FP ghost terms and find the masses of the FP ghosts for general ξ. Compare them to the masses of the scalar field fluctuations obtained in part a).

Solution: Applying the transforms in (22.82) to the gauge fixing functions and keeping only the v terms:

$$\Delta(\partial \cdot B + i\xi g_1(\hat{\phi}^\dagger v - v^\dagger \hat{\phi})/2) \quad \sim \quad \partial^2\eta - \frac{\xi g_1 v^2}{2}[g_1\eta - g_2\theta^3] \tag{9}$$

$$\Delta[\partial \cdot W^a - ig_2\xi(\hat{\phi}^\dagger T_a v - v^\dagger T_a \hat{\phi})] \quad \sim \quad \partial^2\theta^a - \frac{\xi g_2 v^2}{2}\begin{cases}(-g_1\eta + g_2\theta^3) & a=3 \\ g_2\theta^a & a=1,2\end{cases} \tag{10}$$

The cases $a = 1, 2$ of the last equation shows that the mass squared of the FP ghost associated with $W^{1,2}$ is $\xi g_2^2 v^2/2 = \xi M_W^2$, matching the mass of $\hat{\phi}^{1,2}$. Forming g_1 times the first equation -g_2 times the $a = 3$ case of the second equation gives a right side

$$\partial^2(g_1\eta - g_2\theta^3) - \frac{\xi(g_1^2 + g_1^2)v^2}{2}[g_1\eta - g_2\theta^3] \tag{11}$$

Showing that the mass squared of $(g_1\eta - g_2\theta^3)$ is ξM_Z^2. Taking g_2 times the first plus g_1 times the $a = 3$ case of the second shows that the mass of the orthogonal combination $g_2\eta + g_1\theta^3$ is zero matching that of the photon.

17. **Decay of the W.**

a) Calculate the decay rate for the processes $W^- \to l + \bar{\nu}_l$ for each of the leptons $l = e, \mu, \tau$ and $W^- \to D + \bar{U}$ for the quarks $D = d, s, b$ and $U = u, c$. Be sure to include the CKM quark mixing matrix. Why do we leave t out of the possible final states?

Solution: The top quark is heavier than the W. The Feynman amplitude for the lepton processes is

$$\begin{aligned}
\mathcal{M} &= \frac{g_2}{2\sqrt{2}}\bar{u}(p)\gamma \cdot \epsilon(1-\gamma^5)v(p') &(12)\\
\sum_{spins}|\mathcal{M}|^2 &= \frac{g_2^2}{8}\mathrm{Tr}\gamma \cdot \epsilon(1-\gamma_5)(-\gamma \cdot p')\gamma \cdot \epsilon^*(1-\gamma_5)(m - \gamma \cdot p) \\
&= \frac{g_2^2}{4}\mathrm{Tr}\gamma \cdot \epsilon(\gamma \cdot p')\gamma \cdot \epsilon^*(1-\gamma_5)(\gamma \cdot p) \\
&= g_2^2\left[p \cdot \epsilon p' \cdot \epsilon^* + p \cdot \epsilon^* p' \cdot \epsilon - p \cdot p' + i\epsilon^{\kappa\lambda\mu\nu}p_\kappa \epsilon_\lambda p'_\mu \epsilon^*_\nu\right] &(13)
\end{aligned}$$

Let $Q = p + p'$ be the momentum of the W, so $Q \cdot \epsilon = 0$, and hence $p' \cdot \epsilon = -p \cdot \epsilon$. We work in the W rest frame, $\boldsymbol{Q} = 0$. Then we can write the last term as

$$i\epsilon^{\kappa\lambda\mu\nu} p_\kappa \epsilon_\lambda p'_\mu \epsilon^*_\nu = i\epsilon^{\kappa\lambda\mu\nu} p_\kappa \epsilon_\lambda Q_\mu \epsilon^*_\nu = iQ_0 \epsilon^{0kln} p_k \epsilon_l \epsilon^*_n = -iM_W \boldsymbol{p} \cdot (\boldsymbol{\epsilon} \times \boldsymbol{\epsilon}^*) \tag{14}$$

and Then

$$\sum_{spins} |\mathcal{M}|^2 = g_2^2 \left[-2p \cdot \epsilon p \cdot \epsilon^* + \frac{M_W^2 - m_e^2}{2} - iM_W \boldsymbol{p} \cdot (\boldsymbol{\epsilon} \times \boldsymbol{\epsilon}^*) \right] \tag{15}$$

The total rate is given by

$$\begin{aligned} \Gamma &= \int \frac{p^2 dp d\Omega}{(2\pi)^3 8 E_\nu E_e M_W} 2\pi \delta(M_W - E_\nu - E_e) \langle |\mathcal{M}|^2 \rangle \\ &= \int \frac{d\Omega}{(2\pi)^2 8 M_W} \frac{1}{2} \sqrt{1 - \frac{(m_1 + m_2)^2}{M_W^2}} \sqrt{1 - \frac{(m_1 - m_2)^2}{M_W^2}} \langle |\mathcal{M}|^2 \rangle \end{aligned} \tag{16}$$

Here p is determined from energy conservation

$$\sqrt{p^2 + m_1^2} = M_W - \sqrt{p^2 + \mu_2^2}, \qquad \frac{p}{M_W} = \frac{1}{2} \sqrt{1 - \frac{(m_1 + m_2)^2}{M_W^2}} \sqrt{1 - \frac{(m_1 - m_2)^2}{M_W^2}} \tag{17}$$

For the lepton case

$$\int d\Omega \langle | \rangle \mathcal{M}|^2 = 4\pi g_2^2 \left[\frac{M_W^2 - m_e^2}{2} - \frac{2}{3} p^2 \right] = 4\pi g_2^2 (M_W^2 - m_e^2) \left[\frac{1}{3} + \frac{m_e^2}{6M_W^2} \right] \tag{18}$$

so the total rate is

$$\Gamma_{l\nu_l} = \frac{g_2^2 M_W^3}{48\pi M_W^2} \left(1 - \frac{m_e^2}{M_W^2} \right)^2 \left(1 + \frac{m_e^2}{2M_W^2} \right) = \frac{G_F M_W^3}{6\pi\sqrt{2}} \left(1 - \frac{m_e^2}{M_W^2} \right)^2 \left(1 + \frac{m_e^2}{2M_W^2} \right) \tag{19}$$

For the quarks, if we keep in the masses, $\langle |\mathcal{M}|^2 \rangle$ has an extra term proportional to $m_i m_j$, the phase space is more complicated, there are three colors of each flavor, and a factor $|V_{ij}|^2$ where V is the Kobayashi-Maskawa flavor changing matrix. These differences then lead to

$$\Gamma_{ij} = 3|V_{ij}|^2 \frac{G_F M_W^3}{6\pi\sqrt{2}} \sqrt{1 - \frac{(m_i + m_j)^2}{M_W^2}} \sqrt{1 - \frac{(m_i - m_j)^2}{M_W^2}} \left[1 - \frac{m_i^2 + m_j^2}{2M_W^2} - \frac{(m_i^2 - m_j^2)^2}{2M_W^2} \right]$$

b) We cannot observe the quark final states, since they are confined inside of hadrons. Instead the produced quarks will be 100% converted into hadrons. Assuming that the total rate for decay into all possible hadronic states is given by the total rate calculation for all possible quark anti-quark pairs, estimate the ratio $\Gamma_{W\to hadrons}/\Gamma_{W\to leptons}$ and compare to the data. You may neglect the allowed quark and lepton masses in comparison with M_W.

Solution: If we neglect masses, all of the decay rates simplify to

$$\Gamma_{l\nu_l} = \frac{g_2^2 M_W^3}{48\pi M_W^2} = \frac{G_F M_W^3}{6\pi\sqrt{2}} \qquad \Gamma_{ij} = 3|V_{ij}|^2 \frac{G_F M_W^3}{6\pi\sqrt{2}} \tag{20}$$

The total leptonic rate is simply a factor of 3 times the first expression. The total hadronic rate is a bit more subtle since we sum over 3 down type quarks but only over two up type quarks. This involves the CKM matrix:

$$\sum_{j=u,c}\sum_{i=d,s,b} V^{\dagger}_{ji} V_{ij} = \sum_{j=u,c} \delta_{jj} = 2 \tag{21}$$

then

$$\frac{\Gamma_{\text{hadrons}}}{\Gamma_{\text{leptons}}} \approx \frac{3 \cdot 2}{3} = 2 \tag{22}$$

Looking up the data, the branching fraction into hadrons is $0.6741 \pm .0027$ and that into leptons is 0.3086 ± 0.0009. The ratio is $0.6741/0.3086 \approx 2.18 \pm .02$. Pretty good agreement for such a simple model!