Standard Model/Quantum Field Theory

Solution Set 6

Due: 15 November 2019

Suggested reading: QFT Notes Ch 23-24; Textbook: Secs. 83-84.

18. Suppose for a particular definition of the renormalized coupling constant, the power series for $\beta(g)$ is known

$$\beta(g) \quad = \quad b_1 g^3 + b_2 g^5 + b_3 g^7 + \cdots \tag{1}$$

In class we showed that with a redefinition of the renormalized coupling given as a power series

$$g'(g) = g + a_1 g^3 + a_2 g^5 + a_3 g^7 + \cdots , \tag{2}$$

the new beta function $\beta'(g')$ has a power series expansion where the first two terms $b_1 g'^3 + b_2 g'^5$ have coefficients identical to the old beta function.

a) Show, however, that the coefficient of g'^7 is changed, and further that there is a choice of the a_i which sets it to zero.

Solution: Write

$$g'(g) \quad = \quad \sum_{n=0}^{\infty} a_n g^{2n+1}, \qquad a_0 = 1 \tag{3}$$

$$\beta(g) \quad = \quad \sum_{n=1}^{\infty} b_n g^{1+2n} \tag{4}$$

It is more convenient to write the first equation as

$$g(g') \quad = \quad \sum_{n=0}^{\infty} a'_n g'^{2n+1} = g' \sum_{n=0}^{\infty} a'_n g'^{2n}, \qquad a'_0 = 1 \tag{5}$$

$$\tag{6}$$

Then

$$\begin{aligned} \beta'(g') \quad &\equiv \quad \mu \frac{d}{d\mu} g'(g) = \beta(g) \frac{dg'}{dg} = \beta(g(g')) \frac{1}{1 + 3a'_1 g'^2 + \sum_{n=2}^{\infty}(2n+1) a'_n g'^{2n}} \\ &= \quad \sum_{m=1}^{\infty} b_m g'^{2m+1} \frac{\left(1 + \sum_{n=1}^{\infty} a'_n g'^{2n}\right)^{2m+1}}{1 + 3a'_1 g'^2 + \sum_{n=2}^{\infty}(2n+1) a'_n g'^{2n}} \end{aligned} \tag{7}$$

By inspection of this formula we see that the $m = 1$ term is $b_1 g'^3(1 + O(g'^4)$ and that the $m = 2$ term is $b_2 g'^5(1 + O(g'^2)$, which each lead to a modification of the $m = 7$ term. We find in the first case

$$\frac{\left(1 + \sum_{n=1}^{\infty} a'_n g'^{2n}\right)^3}{1 + 3a'_1 g'^2 + \sum_{n=2}^{\infty}(2n+1) a'_n g'^{2n}} \sim \frac{\left(1 + 3(a'_1 g'^2 + a'_2 g'^4) + 3a'^2_1 g'^4\right)}{1 + 3a'_1 g'^2 + 5a'_2 g'^4} \sim 1 + (3a'^2_1 - 2a'_2) g'^4 \tag{8}$$

and in the second case

$$\frac{\left(1+\sum_{n=1}^{\infty} a'_n g'^{2n}\right)^5}{1+3a'_1 g'^2+\sum_{n=2}^{\infty}(2n+1)a'_n g'^{2n}} \sim 1+2a'_1 g'^2 \tag{9}$$

Thus the coefficient of g'^7 in β' is

$$b_3+2a'_1 b_2+(3a'^2_1-2a'_2)b_1 \tag{10}$$

and there are many choices for a'_1 and a'_2 to mke this zero, the simplest is $a'_1 = 0$ and $a'_2 = b_3/(2b_1)$.

b) Argue that the a_i can be chosen so that the coefficients of all terms beyond the first two in β' are zero. In other words, there is a definition of the renormalized coupling such that the two loop beta function is the complete beta function! This was first noted by 't Hooft.

Solution: By a) we can assume $b_3 = 0$ and seek to cancel b_5. To keep $b_3 = 0$, when we repeat the procedure, we assume $a'_1 = a'_2 = 0$. Then the $m = 1$ term is

$$b_1 g'^3 \frac{\left(1+3a'_3 g'^6\right)}{1+7a'_3 g'^6} \sim b_1 g'^3(1-4a'_s g'^6) \tag{11}$$

and the modification from the $m = 2$ term starts at order g'^{11}. Thus a'_3 can be chosen to cancel the $m = 4$ term. The proof proceeds by induction. Assuming $b_m = 0$ for $m = 3, \ldots, N$ modify the coupling with $a'_n = 0$ for $n = 1, \ldots, N-1$, and choose a'_N to cancel the $m = N+1$ term.

c) Assuming this has been done, solve the renormalization group equation exactly to get $t(g)$ as a function of g. Invert it (iteratively as $t \to \infty$ to get $g(t)$ as a function of t keeping terms of order $1/t^3$ and $(\ln t)/t^2$. as $t \to \infty$.

Solution: According to the RG

$$\begin{aligned}
t &= \int_{g_0}^{g(t)} \frac{dg'}{b_1 g'^3+b_2 1 g'^5} = \frac{1}{2b_1}\int_{g(t)^{-2}}^{g_0^{-2}} dv \frac{v}{v+b_2/b_1} \\
2b_1 t &= g_0^{-2} - g^{-2} - \frac{b_2}{b_1}\ln\frac{g_0^{-2}+b_2/b_1}{g^{-2}+b_2/b_1} \\
g^2 &= \frac{1}{g_0^{-2}-2b_1 t-(b_2/b_1)[\ln(g^2/g_0^2)+\ln(b_1+b_2 g_0^2)/(b_1+b_2 g^2)]} \\
&= \frac{1}{g_0^{-2}-2b_1 t}\left[1+\frac{(b_2/b_1)}{g_0^{-2}-2b_1 t}[-\ln(g_0^{-2}-2b_1 t)+\ln(g_0^{-2}+b_2/b_1)]\right]+O(t^{-3}\ln^2 t)
\end{aligned}$$

19. The result we quoted for the QCD beta function (Notes Eq(23.10)

$$\beta(g) = -\frac{Ng^3}{16\pi^2}\left(\frac{11}{3}-\frac{2N_f}{3N}\right) \tag{12}$$

includes the effects of the quark loop on the gluon propagator (N is the number of colors (3 for QCD), and N_f is the number of quark flavors (6) to date). We did not do this part of the calculation in class. By adapting the QED vacuum polarization calculation to QCD, complete the calculation of this beta function and confirm the correctness of the quoted result.

Solution: The beta function for QED is related to the Gell-Mann-Low function $\psi(e^2)$ by

$$\psi(e^2) = \mu \frac{de^2}{d\mu} = 2e\beta(e) \tag{13}$$

Then we read off $\beta(e)$ at one loop:

$$\beta_{1\ loop}(e) = \frac{1}{2e}\frac{e^4}{6\pi^2} = \frac{e^3}{12\pi^2} \tag{14}$$

To fashion this as the quark loop contribution to the QCD beta function, we note that the QCD calculation of the one quark loop diagram compared to the QED calculation replaces e^2 by $g^2 N_f \mathrm{Tr} t_a t_b = g^2 N_f \delta_{ab}/2$. Thus the contribution to β_{QCD} is

$$\frac{g^3 N_f}{24\pi^2} = \frac{Ng^2}{16\pi^2}\frac{2N_f}{3N} \tag{15}$$

which matches the corresponding term in our notes Eq. (23.10).

20. Confirm our expressions for $\gamma_G, \gamma_q, \beta$ in Notes section 23.2 for the gauge $\xi = 0$, by plugging our Landau gauge one loop evaluations of $G^{2,0}$ (gluon propagator), $G^{0,2}$ (quark propagator), and $G^{1,2}$ quark gluon vertex, Notes section 21.5, into the Callan-Symanzik equation.

Solution: The 1PIR one loop correction to the gluon propagator in Landau gauge is given in Eq. (21.60), where we replace μ by a momentum scale Q. Then

$$\begin{aligned} G^{2,0}(Q) &= \frac{-i}{q^2}\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\left[1 + \frac{26C_A}{12}\frac{g^2}{8\pi^2}\ln\frac{\Lambda}{Q}\right] \\ G_r^{2,0} &= \frac{-i}{q^2}\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\left[1 + \frac{26C_A}{12}\frac{g^2}{8\pi^2}\ln\frac{\mu}{Q}\right] \qquad (16) \\ \mu\frac{\partial}{\partial\mu}G_r^{2,0} &= \frac{26C_A}{12}\frac{g^2}{8\pi^2}G^{2,0} + O(g^4) \qquad (17) \end{aligned}$$

The C-G equation in Landau gauge is

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} - n\gamma_G - m\gamma_q\right]G_r^{n,m} = 0. \tag{18}$$

Putting $m = 0$ and $n = 2$, and noting that the derivative w.r.t. g contributes at order g^4, we conclude that

$$\gamma_G(g) = \frac{13C_A}{6}\frac{g^2}{16\pi^2} + O(g^4) \tag{19}$$

in agreement with Eq (23.8) for $\xi = 0$ and $N_f = 0$. In Landau gauge the massless quark propagator is independent of μ, so $\gamma_q = 0 + O(g^4)$ in agreement with Eq. (23.9) for $\xi = 0$. Finally the quark gluon vertex (with gluon propagator attached) is given by combining (21.56) with (21.60) Plus the zeroth order vertex:

$$
\begin{aligned}
G^{1,2} &= ig\gamma^\mu T_a \left[1 + \frac{3g^2C_A}{32\pi^2}\ln\frac{\Lambda}{Q} + \frac{26C_A}{12}\frac{g^2}{8\pi^2}\ln\frac{\Lambda}{Q}\right] \\
G_r^{1,2} &= ig\gamma^\mu T_a \left[1 + \frac{3g^2C_A}{32\pi^2}\ln\frac{\mu}{Q} + \frac{26C_A}{12}\frac{g^2}{8\pi^2}\ln\frac{\mu}{Q}\right] \\
\mu\frac{\partial}{\partial\mu}G_2^{1,2} &= \left[\frac{3g^2C_A}{32\pi^2} + \frac{26C_A}{12}\frac{g^2}{8\pi^2}\right]G_2^{1,2} \qquad (20)
\end{aligned}
$$

According to the C-S equation the right side should be

$$
\begin{aligned}
\left[-\beta\frac{\partial}{\partial g} + \gamma_G\right]G^{1,2} &= \left[-\frac{\beta}{g} + \gamma_G\right]G^{1,2} + O(g^5) \\
-\frac{\beta}{g} &= \frac{3g^2C_A}{32\pi^2} + \frac{26C_A}{12}\frac{g^2}{8\pi^2} - \gamma_G = \frac{g^2C_A}{16\pi^2}\left[\frac{3}{2} + \frac{13}{6}\right] = \frac{11}{3}\frac{g^2C_A}{16\pi^2} \\
\beta(g) &= -\frac{11}{3}\frac{g^3N}{16\pi^2} \qquad (21)
\end{aligned}
$$

in agreement with (23.10) for $N_f = 0$. $N_f \neq 0$ is discussed in the previous problem.

21. **Muon Decay** The detailed study of the process $\mu^- \to e^- + \nu_\mu + \bar{\nu}_e$ was crucial to establishing the current-current structure of the weak interactions, but in modern terms it represents the cleanest measurement of G_F.

a) Starting with the Feynman rules of the Standard Model Lagrangian, show that the tree amplitude for this process is, to a very good approximation (note that all momenta are of $O(.1\text{GeV}) \ll M_W$!)

$$
\mathcal{M} = i\frac{G_F}{\sqrt{2}}\bar{u}_{\nu_\mu}\gamma^\lambda(1-\gamma_5)u_\mu\bar{u}_e\gamma_\lambda(1-\gamma_5)v_{\bar{\nu}_e} \qquad (22)
$$

Solution: The terms in the standard model Lagrangian contributing to this process are

$$
-ig_2\bar{L}_e\frac{\tau_1W_1 + \tau_2W_2}{2}\frac{1-\gamma_5}{2}L_e - ig_2\bar{L}_\mu\frac{\tau_1W_1 + \tau_2W_2}{2}\frac{1-\gamma_5}{2}L_\mu \qquad (23)
$$

where $L_{e,\mu}$ are the electron and muon SU(2) doublets. The combination

$$
\frac{\tau_1W_1 + \tau_2W_2}{2} = \frac{1}{2}\begin{pmatrix} 0 & W_1 - iW_2 \\ W_1 + iW_2 & 0 \end{pmatrix} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix} \qquad (24)
$$

So we can write the relevant terms in the Lagrangian as

$$
\frac{-ig_2}{2\sqrt{2}}\left[\bar{\nu}_e\gamma\cdot W(1-\gamma_5)e + \bar{\nu}_\mu\gamma\cdot W(1-\gamma_5)\mu + \bar{\mu}\gamma\cdot W^\dagger(1-\gamma_5)\nu_\mu + \bar{e}\gamma\cdot W^\dagger(1-\gamma_5)\nu_e\right] \qquad (25)
$$

The second and last terms contribute to muon decay:

$$
\begin{aligned}
\mathcal{M} &= \frac{ig_2^2}{8}\bar{u}_{\nu_\mu}\gamma^\lambda(1-\gamma_5)u_\mu\bar{u}_e\gamma_\kappa(1-\gamma_5)v_{\bar{\nu}_e}\frac{\eta_{\kappa\lambda} + q_\kappa q_\lambda/M_W^2}{q^2 + M_W^2} \\
&\approx \frac{iG_F}{\sqrt{2}}\bar{u}_{\nu_\mu}\gamma^\lambda(1-\gamma_5)u_\mu\bar{u}_e\gamma_\kappa(1-\gamma_5)v_{\bar{\nu}_e} \qquad (26)
\end{aligned}
$$

where we used $M_W \gg q$ and the definition $G_F/\sqrt{2} = g_2^2/(8M_W^2)$.

b) Since the neutrinos are unobservable, it is convenient to rearrange the spinors in this expression so that the neutrino variables are in the same factor. Prove the Fierz rearrangement identity:

$$[\gamma^\mu(1-\gamma_5)]_{\alpha\beta}[\gamma_\mu(1-\gamma_5)]_{\gamma\delta} = -[\gamma^\mu(1-\gamma_5)]_{\alpha\delta}[\gamma_\mu(1-\gamma_5)]_{\gamma\beta}. \tag{27}$$

Hint: use the fact that any 4×4 matrix can be written as a linear combination of the 16 matrices $I, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}$. Then the neutrino phase space integral involves

$$\int \frac{d^3q_1 d^3q_2}{4|q_1||q_2|(2\pi)^6}\delta^4(q_1+q_2+Q)\mathrm{Tr} q_1\cdot\gamma\gamma_\lambda(1-\gamma_5)q_2\cdot\gamma\gamma_\kappa(1-\gamma_5) = N[Q_\lambda Q_\kappa - \eta_{\kappa\lambda}Q^2] \tag{28}$$

where $Q = p_e - p_\mu$. Prove this formula, find N, and evaluate $|\mathcal{M}|^2$ for a *polarized* μ, integrated over the neutrino phase space and summed over the spin of the electron in the final state. Evaluate the differential rate $d^2\Gamma/(dE_e d\Omega)$, where E_e is the electron energy in the muon's rest frame.

Solution: For the Fierz relation we write an expansion

$$[\gamma^\mu(1-\gamma_5)]_{\alpha\beta}[\gamma_\mu(1-\gamma_5)]_{\gamma\delta} = I_{\alpha\delta}C^1_{\gamma\beta} + \gamma_{g\alpha\delta}C^2_{\gamma\beta} + \gamma^\mu_{\alpha\delta}C^3_{\mu\gamma\beta} + (\gamma^\mu\gamma_5)_{\alpha\delta}C^4_{\mu\gamma\beta} + \sigma^{\mu\nu}_{\alpha\delta}C^5_{\mu\nu\gamma\beta} \tag{29}$$

Tracing both sides on $\alpha\delta$ shows that $C^1 = 0$. Multiplying both sides with $\gamma_{5\delta\alpha}$ and summing α, δ shows that $C^2 = 0$, and doing the same with $\sigma^{\mu\nu}$ shows that $C^5 = 0$. So we have reduced the identity to

$$[\gamma^\mu(1-\gamma_5)]_{\alpha\beta}[\gamma_\mu(1-\gamma_5)]_{\gamma\delta} = +\gamma^\mu_{\alpha\delta}C^3_{\mu\gamma\beta} + (\gamma^\mu\gamma_5)_{\alpha\delta}C^4_{\mu\gamma\beta} \tag{30}$$

Multiplying both sides by $\gamma_{\lambda\delta\alpha}$ and summing gives

$$[\gamma_\mu(1-\gamma_5)\gamma_\lambda\gamma^\mu(1-\gamma_5)]_{\gamma\beta} = -4C^3_{\lambda\gamma\beta} \tag{31}$$

From which we find $C^3_{\lambda\gamma\beta} = -(\gamma_\lambda(1-\gamma_5)_{\gamma\beta}$. Finally do the same with $(\gamma_\lambda\gamma_5)_{\delta\alpha}$:

$$[\gamma_\mu(1-\gamma_5)\gamma_\lambda\gamma_5\gamma^\mu(1-\gamma_5)]_{\gamma\beta} = \mathrm{Tr}\gamma_\lambda\gamma_5\gamma^\mu\gamma_5 C^4_{\mu\gamma\beta} = 4C^4_{\lambda\gamma\beta} \tag{32}$$

which shows that $C^4_{\lambda\gamma\beta} = (\gamma_\lambda(1-\gamma_5))_{\gamma\beta}$ and the identity is proven.

Using the Fierz rearrangement on the expression for $\mathcal{M}$ gives

$$\mathcal{M} = -i\frac{G_F}{\sqrt{2}}\bar{u}_{\nu_\mu}\gamma^\lambda(1-\gamma_5)v_{\bar{\nu}_e}\bar{u}_e\gamma_\lambda(1-\gamma_5)u_\mu \tag{33}$$

Writing the square of the neutrino matrix element as a trace, we see that the integral over neutrino phase space involves the integral quoted above. The integral is a Lorentz covariant tensor of rank 2, and multiplying it by Q^λ and summing λ gives zero (because $q_1^2 = q_2^2 = 0$. Thus it must be of the form $N[Q_\lambda Q_\kappa - \eta_{\lambda\kappa}Q^2]$. To evaluate N take the trace of both sides

$$\begin{aligned}
-3Q^2N &= \int \frac{d^3q_1 d^3q_2}{4|q_1||q_2|(2\pi)^6}\delta^4(q_1+q_2+Q)4\mathrm{Tr}q_1\cdot\gamma q_2\cdot\gamma(1-\gamma_5) \\
&= -16\int \frac{d^3q_1 d^3q_2}{4|q_1||q_2|(2\pi)^6}\delta^4(q_1+q_2+Q)q_1\cdot q_2 \\
&= = -8Q^2\int \frac{d^3q_1 d^3q_2}{4|q_1||q_2|(2\pi)^6}\delta^4(q_1+q_2+Q)
\end{aligned} \tag{34}$$

Evaluating the integral in a frame where $\boldsymbol{Q} = 0$ gives

$$3N = \frac{2}{(2\pi)^6} 4\pi \int_0^\infty dq_1 \delta(2q_1 + Q^0) = \frac{2}{(2\pi)^5}\theta(-Q^0) \tag{35}$$

The square of the lepton factor summed over electron spins, but not over muon spins, times the phase space integral of the neutrino squared amplitude gives the differential rate:

$$\begin{aligned} d\Gamma &= \frac{G_F^2}{2}\frac{d^3p_e}{(2\pi)^3 4E_e m_\mu}\bar{u}_\mu \gamma^\kappa(1-\gamma_5)(m_e - \gamma\cdot p_e)\gamma^\lambda(1-\gamma_5)u_\mu \frac{Q_\kappa Q_\lambda - Q^2\eta_{\kappa\lambda}}{3\pi} \\ &= -2\frac{G_F^2}{2}\frac{d^3p_e}{(2\pi)^3 4E_e m_\mu}\bar{u}_\mu \gamma^\kappa \gamma\cdot p_e \gamma^\lambda(1-\gamma_5)u_\mu \frac{Q_\kappa Q_\lambda - Q^2\eta_{\kappa\lambda}}{3\pi} \end{aligned} \tag{36}$$

Next we use $Q = p_e - p_\mu$ to rewrite

$$\bar{u}_\mu Q\cdot\gamma\gamma\cdot p_e Q\cdot\gamma(1-\gamma_5)u_\mu = Q^2\bar{u}_\mu\gamma\cdot p_e)(1-\gamma_5)u_\mu - 2p_e\cdot Q\bar{u}_\mu Q\cdot\gamma(1-\gamma_5)u_\mu \tag{37}$$

$$-Q^2\bar{u}_\mu\gamma_\lambda\gamma\cdot p_e\gamma^\lambda(1-\gamma_5)u_\mu = -2Q^2\bar{u}_\mu\gamma\cdot p_e(1-\gamma_5)u_\mu \tag{38}$$

In the muon rest frame $u_\mu = \sqrt{2m_\mu}(\phi, 0)$ and

$$\bar{u}_\mu\gamma\cdot p_e(1-\gamma_5)u_\mu = 2m_\mu(-\boldsymbol{p}_e\cdot\phi^\dagger\boldsymbol{\sigma}\phi - E_e) \tag{39}$$

$$\bar{u}_\mu\gamma\cdot Q(1-\gamma_5)u_\mu = 2m_\mu(-\boldsymbol{p}_e\cdot\phi^\dagger\boldsymbol{\sigma}\phi - (E_e - m_\mu)) \tag{40}$$

$$Q^2 = -m_e^2 - m_\mu^2 + 2E_e m_\mu, \qquad -2p_e\cdot Q = 2m_e^2 - 2E_e m_\mu \tag{41}$$

Inserting these results in $d\Gamma$,

$$d\Gamma = G_F^2\frac{d^3p_e}{48\pi^4 E_e}\left[Q^2(-\boldsymbol{p}_e\cdot\phi^\dagger\boldsymbol{\sigma}\phi - E_e) + 2p_e\cdot Q(-\boldsymbol{p}_e\cdot\phi^\dagger\boldsymbol{\sigma}\phi - (E_e - m_\mu))\right] \tag{42}$$

$$\frac{d^2\Gamma}{dp_e d\Omega} = \frac{G_F^2 p_e^2}{48\pi^4 E_e}\left[3m_\mu^2 E_e - 4m_\mu E_e^2 + 3m_e^2 E_e - 2m_\mu m_e^2 - \boldsymbol{p}_e\cdot\langle\boldsymbol{\sigma}\rangle(4m_\mu E_e - m_\mu^2 - 3m_e^2)\right]$$

where $\langle\boldsymbol{\sigma}\rangle = \bar{u}_\mu\boldsymbol{\Sigma}u_\mu/\bar{u}_\mu u_\mu = \phi^\dagger\boldsymbol{\sigma}\phi$ is the muon polarization in its rest frame.

c) Explain what feature of this distribution implies parity violation. How does the distribution change for the charge conjugated process $\mu^+ \to e^+ + \bar{\nu}_\mu + \nu_e$?

Solution:The term $-\boldsymbol{p}_e\cdot\langle\boldsymbol{\sigma}\rangle$, which indicates a preferred electron emission anti-parallel to the muon polarization, violates parity under which $\boldsymbol{p}_e$ changes sign but $\langle\boldsymbol{\sigma}\rangle$ remains invariant. For the charge conjugate process, the parity violating term has the opposite sign so positron emission is preferentially parallel to the anti-muon polarization. The result is invariant under the combined CP sym=mmetry.

d) Calculate the total rate assuming $m_e = 0$. By comparing this to the observed rate, find the value for G_F implied by this tree approximation.

Solution:Integration over $d\Omega$ kills the $\boldsymbol{p}_e\cdot\langle\boldsymbol{\sigma}\rangle$ and multiplies the rest by 4π:

$$\Gamma = \int dp_e \frac{G_F^2 p_e}{12\pi^3}\left[3m_\mu^2 p_e - 4m_\mu p_e^2\right] \tag{43}$$

where we also set $m_e = 0$. The range of p_e is determined by $p_e + q_1 + q_2 = m_\mu$ where $\boldsymbol{p}_e = -\boldsymbol{q}_1 - \boldsymbol{q}_2$. It is maximal $m_\mu/2$ when the $\boldsymbol{q}$'s are parallel and 0 when they are equal and opposite. Then

$$\begin{aligned} \Gamma &= \int_0^{\mu_\mu/2} dp_e \frac{G_F^2}{12\pi^3}\left[3m_\mu^2 p_e^2 - 4m_\mu p_e^3\right] = \frac{G_F^2 m_\mu^5}{192\pi^3} \\ G_F &= \sqrt{\frac{192\pi^2\Gamma}{m_\mu^5}} \end{aligned} \tag{44}$$