Standard Model/Quantun Field Theory

Solution Set 7

Due: 4 December 2019

Suggested reading: QFT Notes Ch 25; Textbook: Sec. 83

22. **The Higgs particle**. As we have seen in class, the way particles are given mass in the standard model is through coupling to the complex scalar Higgs field doublet ϕ which has a vacuum expectation value which we describe by writing

$$\phi = \begin{pmatrix} v + (h + ia)/\sqrt{2} \\ \phi_2 \end{pmatrix} \tag{1}$$

where we take the vacuum value v to be a real number, and h and a are real fields. The complex field ϕ_2 and the real field a provide the zero helicity fields for the massive vector bosons W and Z. The remaining real field h describes a new massive spin zero particle, the Higgs particle. Its mass M_h and its quartic self coupling λ are independent parameters in the standard model.

a) Identify the cubic terms in the standard model Lagrangian that contain one factor of h and two other fields: two vector bosons, quark-antiquark, and lepton-antilepton. Hint: since h enters the Lagrangian with v in the combination $v+h/\sqrt{2}$, you can find these terms by substituting $v \to v + h/\sqrt{2}$ in the mass (quadratic) terms of the Lagrangian and expanding to first order in h. Use the fact that the particle masses are all proportional to v.

Solution: The vector boson mass terms are

$$\begin{aligned} \frac{M_Z^2}{2} Z^2 + M_W^2 W^\dagger \cdot W &= \frac{g_2^2 v^2}{2} W^\dagger \cdot W + \frac{g_2^2 v^2}{4\cos^2\theta_W} Z^2 \\ &\to \frac{g_2^2 (v + h/\sqrt{2})^2}{2} W^\dagger \cdot W + \frac{g_2^2 (v + h/\sqrt{2})^2}{4\cos^2\theta_W} Z^2 \\ &\to \frac{M_Z^2}{2} Z^2 + M_W^2 W^\dagger \cdot W + h\left[\frac{g_2^2 v\sqrt{2}}{2} W^\dagger \cdot W + \frac{g_2^2 v \sqrt{2}}{4\cos^2\theta_W} Z^2\right] \\ &= \frac{M_Z^2}{2} Z^2 + M_W^2 W^\dagger \cdot W + h\left[g_2 M_W W^\dagger \cdot W + \frac{g_2 M_W}{2\cos^2\theta_W} Z^2\right] \end{aligned} \tag{2}$$

where we eliminated $v = M_W\sqrt{2}/g_2$ in the last line. The fermion mass terms are linear in v:

$$m_f \bar{f} f = G_f v \bar{f} f \to m_f \bar{f} f + h \frac{G_f}{\sqrt{2}} \bar{f} f = m_f \bar{f} f + h \frac{g_2 m_f}{2M_W} \bar{f} f \tag{3}$$

In summary the cubic terms in the Lagrangian are:

$$-h\left(g_2 M_W W^\dagger \cdot W + \frac{g_2 M_Z}{2\cos\theta_W} Z \cdot Z + \sum_f \frac{g_2 m_f}{2M_W} \bar{f} f\right) \tag{4}$$

b) Extract the Feynman rules for the cubic vertices corresponding to each of these terms.

Solution: We simply read off:

$$-ig_2 M_W, \qquad -i\frac{g_2 M_Z}{\cos\theta_W}, \qquad -i\frac{g_2 m_f}{2M_W} \tag{5}$$

for the three terms respectively.

c) Calculate the decay rate into each of these two body final states, (assuming M_h is large enough for the decay to proceed in each case). Now that the Higgs particle mass is known to be $M_h \approx 125$GeV, the decay into a pair of vector bosons is purely an academic exercise!

Solution:

$$\Gamma_{h\to WW} = \frac{g_2^2 M_h^3}{64\pi M_W^2}\left[1-4\frac{M_W^2}{M_h^2}+12\frac{M_W^4}{M_h^4}\right]\sqrt{1-4\frac{M_W^2}{M_h^2}} \tag{6}$$

$$\Gamma_{h\to ZZ} = \frac{g_2^2 M_h^3}{128\pi M_Z^2\cos^2\theta_W}\left[1-4\frac{M_Z^2}{M_h^2}+12\frac{M_Z^4}{M_h^4}\right]\sqrt{1-4\frac{M_Z^2}{M_h^2}} \tag{7}$$

$$\Gamma_{h\to f\bar f} = \frac{g_2^2 M_h m_f^2}{32\pi M_W^2}\left(1-4\frac{m_f^2}{M_h^2}\right)^{3/2} \tag{8}$$

When the fermion is a quark, the corresponding rate is a factor of $N_c = 3$ larger because they come in three colors.

23. **Calculating the effective potential.** Consider the field theory of a single real scalar field ϕ with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4!}(\phi^2 - a^2)^2, \tag{9}$$

which is invariant under $\phi \to -\phi$. Define the action $S(\phi) \equiv \int d^4x\mathcal{L}$. Recall that the effective action $\Gamma(\varphi) \equiv W(J) - \int d^4x\varphi J$, where $e^{iW(J)}$ is the functional integral over ϕ of $\exp\{iS(\phi) + i\int d^4x\phi J\}$, and $\varphi(x) \equiv \langle\phi\rangle_J = \delta W/\delta J$. For the effective potential it will be enough to introduce a constant source J, so $W(J) = VTw(J)$, and $\varphi = \partial w/\partial J$, and $\Gamma(\varphi) = -VTV_{eff}(\varphi)$.

a) In class we have observed that $W(J)$ is the sum of all connected vacuum graphs (0-point functions) for the action $S(\phi)+\int d^4x\phi J$. By changing variables in the path integral for $e^{i\Gamma(\varphi)}$, show that $\Gamma(\varphi)$ is the sum of all connected vacuum graphs for the action $S(\phi+\varphi)+\int d^4x\phi J$, where *$J$ is chosen so that* $\langle\phi\rangle_J = 0$.

Solution: By definition

$$e^{i\Gamma(\varphi)} = e^{iW(J)-i\int d^4x\varphi J} = \int D\phi e^{i\int d^4x[\mathcal{L}+J(\phi-\varphi)]}, \qquad \varphi = \langle\phi\rangle_J \tag{10}$$

Now change path integral variables $\phi = \varphi + \hat\phi$:

$$e^{i\Gamma(\varphi)} = \int D\hat\phi e^{i\int d^4x[\mathcal{L}(\hat\phi+\varphi)+J\hat\phi}, \qquad \langle\hat\phi\rangle_J = 0 \tag{11}$$

But this is what was to be proved: The logarithm of this path integral is $i\times$ the sum of all connected vacuum diagrams, and the condition $\langle\hat{\phi}\rangle_J = 0$ places the desired constraint on the choice of J.

b) In terms of Feynman graphs $\langle\phi(x)\rangle$ is the sum of all connected one-point diagrams ("*tadpoles*") which have a propagator emerging from the point x with the other end hooked to an arbitrary structure with no further external legs. The condition $\langle\phi\rangle_J = 0$ means that the sum of all such tadpole diagrams is zero. But it also means that all diagrams with one or more tadpole subdiagrams will sum to zero. Show that a connected vacuum diagram that is one particle reducible (i.e. can be disconnected by cutting a single line) always has at least one tadpole subdiagram, implying that all one particle reducible diagram contributions to Γ cancel out in the complete sum, so Γ can be calculated as the sum of all connected 1PIR vacuum diagrams.

Solution: Vacuum diagrams have no external legs. A vacuum diagram that can be disconnected by cutting a single internal line has a structure which ends in at least two tadpole subdiagrams each of which have been set to zero by the condition on J. Thus the only nonzero diagrams must be 1PIR.

c) Assuming constant J, φ obtain the Feynman rules for the action $S(\phi+\varphi) + \int d^4x\phi J$. List the propagator, as well as the values of the 1, 3, and 4 point vertices. Note that some of these couplings and the mass occurring in the propagator depend explicitly on φ. Also note that J explicitly appears only in the 1-point vertex. Since this vertex only appears in one particle reducible diagrams the calculation of Γ will have no explicit dependence on J. This means that the complicated relation between φ and J implied by $\langle\phi\rangle_J = 0$ is not needed!

Solution: To get the vertices of this theory, we expand the potential term about $\phi = 0$:

$$\begin{aligned} -\frac{\lambda}{4!}((\phi+\varphi)^2 - a^2)^2 + J\phi &= -\frac{\lambda}{4!}(\varphi^2 - a^2)^2 + \left[J - \frac{\lambda}{3!}\varphi(\varphi^2 - a^2)\right]\phi - \left[\frac{\lambda}{2}\varphi^2 - \frac{\lambda}{6}a^2\right]\frac{\phi^2}{2} \\ &\quad -\frac{\lambda\varphi}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 \end{aligned} \tag{12}$$

defining $m^2(\varphi) = \lambda\varphi^2/2 - \lambda a^2/6$ and $g(\varphi) = \lambda\varphi$ the Lagrangisn becomes

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4!}(\varphi^2 - a^2)^2 + \phi\left[J + \frac{\lambda}{3!}\varphi(a^2 - \varphi^2)\right] - \frac{m^2(\varphi)}{2}\phi^2 - \frac{g(\varphi)}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 \\ \Delta &= \frac{-i}{p^2 + m^2(\varphi)}; \quad V_1 = i\left[J + \frac{\lambda}{3!}\varphi(a^2 - \varphi^2)\right]; \quad V_3 = -ig(\varphi); \quad V_4 = -i\lambda \end{aligned}$$

As noted, V_1 only appears in reducible diagrams, which will vanish once the constraint on J is fulfilled.

d) The zero loop value for $\Gamma(\varphi)$ is clearly just $S(\phi) = -VT(\lambda/4!)(\varphi^2 - a^2)^2$. The one loop vacuum diagram is represented by a diagram with a propagator closing on itself with no vertices, and stands for the result of doing the Gaussian integral over ϕ using the quadratic terms in the action: $\det^{-1/2}(m(\varphi)^2 - \partial^2)$. For constant φ its contribution to $i\Gamma$ is the log, $-(VT/2)\int(d^4p/(2\pi)^4)\ln(m(\varphi)^2 + p^2)$. This diagram is usually ignored when calculating in a source free theory-its just the zero-point energy of the vacuum and has no direct significance.

But for the calculation of the effective potential the mass in the propagator depends on φ and that dependence *is* significant. Calculate the effective potential $V_{eff}(\phi)$ through one loop. Do the one-loop integral in Euclidean space (after Wick rotation) with the simple cutoff $p^2 < \Lambda^2$. Show that the cutoff dependence can be absorbed in renormalized parameters. (We know that there must also be field renomalization $\varphi_r = \varphi/\sqrt{Z}$, but with constant φ you won't be able to separate the part of the divergence to be absorbed in Z).

Solution:

$$i\Gamma_{1\ loop} = -\frac{VT}{2}\int \frac{d^4p}{(2\pi)^4}\ln(p^2+m^2(\varphi)-i\epsilon) = -i\frac{VT}{2}\int \frac{d^4p_E}{(2\pi)^4}\ln(p_E^2+m^2(\varphi)-i\epsilon) \quad (13)$$

after Wick rotation $p^0 = ip^4$. Note that the contour at infinity is infinite, so dropping it means we are actually *defining* the theory in Euclidean space. Then, remembering that the angular integral gives $2\pi^2$,

$$\begin{aligned}
V^{eff}_{1\ loop} &= \frac{1}{16\pi^2}\int_0^\Lambda p^3 dp \ln(p^2+m^2(\varphi)) = \frac{1}{32\pi^2}\int_0^{\Lambda^2} udu\ln(u+m^2(\varphi)) \\
&= \frac{1}{64\pi^2}\Big[\Lambda^4\ln(\Lambda^2+m^2(\varphi)) - \frac{1}{2}\left[(\Lambda^2+m^2(\varphi))^2 - m^4(\varphi)\right] \\
&\quad +2m^2(\varphi)\Lambda^2 - m^4(\varphi)\ln\frac{\Lambda^2+m^2(\varphi)}{m^2(\varphi)}\Big] \\
&\sim \frac{1}{64\pi^2}\left[\Lambda^4\left[\ln\Lambda^2 - \frac{1}{2}\right] + 2m^2(\varphi)\Lambda^2 - m^4(\varphi)(\ln\Lambda^2+1) + m^4(\varphi)\ln m^2(\varphi)\right]
\end{aligned}$$

Since the Λ dependence is a quartic polynomial in φ it can be absorbed in the parameters of the input bare lagrangian. Then we can write the renormalized effective potential through 1 l00p as

$$V^{eff}(\varphi_r) = \frac{\lambda_r}{4!}(\varphi_r^2 - a_r^2)^2 + \frac{\lambda_r^2}{64\pi^2}\left(\frac{\varphi_r^2}{2} - \frac{a_r^2}{6}\right)^2 \ln\left(\frac{\varphi_r^2}{2} - \frac{a_r^2}{6}\right) \quad (14)$$

Note that the ln becomes complex for $\varphi^2 < a_r^2/3$, the inflection point of the input potential.

24. As mentioned in class chiral $SU(2)\times SU(2)$ is isomorphic to $O(4)$. In this exercise we explore the relationship further. Let J_L^i, J_R^i be the commuting generators of the two $SU(2)$'s, rotating the L and R components of the quark fields respectively. They each satisfy the usual angular momentum commutation relations.

a) Show that $J^i = J_L^i + J_R^i$ generates the $O(3)$ rotations of ordinary isospin. That is, $\bar{q}q$, $\bar{q}i\gamma_5 q$ are isoscalars and $\bar{q}\boldsymbol{\tau}q$, $\bar{q}i\gamma_5\boldsymbol{\tau}q$ are isovectors under these transformations.

Solution: Firs note that the $SU(2)$ Lie algebra

$$[J_k, J_l] = [J_k^L, J_l^L] + [J_k^R, J_l^R] = i\epsilon_{klm}(J_m^L + J_m^R) = i\epsilon_{klm}J_m \quad (15)$$

The under J_k since q^L and q^R rotate the same way, $\bar{q}q = \bar{L}R + \bar{R}L$ is invariant, as is $\bar{q}i\gamma_5 q$. Similarly $\bar{q}\boldsymbol{\tau}q$ rotates as an isovector because $\boldsymbol{\tau}$ does..

b) Show that $K^i = J_L^i - J_R^i$ acts as a "boost" that transforms $(\bar{q}q, \bar{q}i\gamma_5\boldsymbol{\tau}q)$ and $(\bar{q}i\gamma_5 q, \bar{q}\boldsymbol{\tau}q)$ as 4-vectors. The terminology here is to suggest an analogy with a Lorentz transformation, but of course $O(4)$ is a compact group. The condensate for isospin conserving chiral symmetry breaking is then $\langle\bar{q}q\rangle = v \neq 0$.

Solution: Let $q \to q - i\epsilon_k(\tau_k/2)q^L + i\epsilon_k(\tau_k/2)q^R = q + i\epsilon_k(\tau_k/2)\gamma_5 q$. Then

$$\begin{aligned}
\bar{q}q &\to \bar{q}q - i\epsilon_a q^\dagger(\tau_a/2)\gamma_5\gamma^0 q + i\epsilon_a q^\dagger(\tau_a/2)\gamma^0\gamma_5 q = \bar{q}q + i\epsilon_a\bar{q}\tau_a\gamma_5 q \\
\bar{q}\tau_b\gamma_5 q &\to \bar{q}q - i\epsilon_a q^\dagger(\tau_a/2)\tau_b\gamma_5\gamma^0\gamma_5 q + i\epsilon_a q^\dagger(\tau_a/2)\gamma^0\tau_b\gamma_5^2 q = \bar{q}\tau_b\gamma_5 q + i\epsilon_a\bar{q}\frac{\{\tau_a,\tau_b\}}{2}q \\
&= \bar{q}\tau_b\gamma_5 q + i\epsilon_b\bar{q}q
\end{aligned} \tag{16}$$

Or

$$\bar{q}q \to \bar{q}q + \epsilon_a\bar{q}\tau_a i\gamma_5 q; \quad \bar{q}i\sigma_a\gamma_5 q \to \bar{q}i\sigma_a\gamma_5 q - \epsilon_b\bar{q}q \tag{17}$$

The four vector with an extra γ_5 works in exactly the same way.

c) Work out the commutators of the $O(4)$ generators J^i, K^i among themselves.

Solution:

$$[J^k, J^l] = i\epsilon_{klm}J^m; \quad [J^k, K^l] = i\epsilon_{klm}K^m; \quad [K^k, K^l] = i\epsilon_{klm}J^m \tag{18}$$