Standard Model/Quantun Field Theory III

Solution Set 1

Due: Wedesday, 22 January 2020

Suggested reading: QFT Notes, Ch 25; Sr, Secs 75-79; P, Ch 19; Sc, Ch 30. Here Sr=Srednicki, P=Peskin&Schroeder, and Sc=Schwartz. These sources cover the same material, but from different poits of view. My notes are self-contained.

1. In our discussion of the axial current anomaly, the triangle diagrams with an axial current at one vertex and vector currents at the other two vertices played a central role. In momentum space this diagram must be proportional to a 3-index psuedotensor $X^{\mu\rho\sigma}(k_1, k_2)$ constructed in terms of the two independent vector momenta k_1^μ, k_2^μ and the Levi-Civita tensor. $\epsilon^{\mu\nu\rho\sigma}$. We found two candidates for X which satisfied Bose statistics in the vector vertices $X^{\mu\rho\sigma}(k_1, k_2) = X^{\mu\sigma\rho}(k_2, k_1)$ and current conservation of the vector currents $k_{1\rho}X^{\mu\rho\sigma} = k_{2\sigma}X^{\mu\rho\sigma} = 0$. Prove that these two candidates are in fact equal to each other. [You may assume that the vector momenta are on photon shell $k_1^2 = k_2^2 = 0$. The offshell case is given in a footnote of my notes.]:

$$\epsilon^{\mu\rho\sigma\tau}(k_1 - k_2)_\tau + \frac{2(k_1 + k_2)^\rho k_{1\tau} k_{2\lambda} \epsilon^{\mu\sigma\lambda\tau}}{(k_1 + k_2)^2} - \frac{2(k_1 + k_2)^\sigma k_{1\tau} k_{2\lambda} \epsilon^{\mu\rho\lambda\tau}}{(k_1 + k_2)^2} = + \frac{2(k_1 + k_2)^\mu k_{1\tau} k_{2\lambda} \epsilon^{\rho\sigma\lambda\tau}}{(k_1 + k_2)^2}. \quad (1)$$

A pedestrian approach to the proof is to work in the rest frame of the two photons $\boldsymbol{k}_1 = -\boldsymbol{k}_2$, in which case you have to go through all the distinct choices of $\mu\rho\sigma$: all space; two space, one time; 1 space, two times; all times. Some of these cases are trivial, but explain why. If you can find a frame independent proof of this identity, by all means present it!

Solution: I give the frame independent proof. Rewrite the identity

$$\epsilon^{\mu\rho\sigma\tau}(k_1 - k_2)_\tau = \frac{2(k_1 + k_2)^\mu k_{1\tau} k_{2\lambda} \epsilon^{\rho\sigma\lambda\tau}}{(k_1 + k_2)^2} - \frac{2(k_1 + k_2)^\rho k_{1\tau} k_{2\lambda} \epsilon^{\mu\sigma\lambda\tau}}{(k_1 + k_2)^2} + \frac{2(k_1 + k_2)^\sigma k_{1\tau} k_{2\lambda} \epsilon^{\mu\rho\lambda\tau}}{(k_1 + k_2)^2}.$$

Next notice that the right side is a three index tensor, completely antisymmetric in its 3 indices. The right side therefore can be written as $\epsilon^{\mu\rho\sigma\tau}v_\tau$ for some four vector v_τ. To figure out v_τ recall the identities

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma'\delta'} = -2(\delta^\gamma_{\gamma'}\delta^\delta_{\delta'} - \delta^\delta_{\gamma'}\delta^\gamma_{\delta'}), \qquad \epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta'} = -6\delta^\delta_{\delta'} \quad (2)$$

We therefore get $-6v_\alpha$ by multiplying the right side by $\epsilon_{\mu\rho\sigma\alpha}$ and summing over repeated indices. The three terms on the right side give equal contributions ao we learn that

$$-6v_\alpha = 6\epsilon_{\mu\rho\sigma\alpha}\frac{(k_1 + k_2)^\mu k_{1\tau} k_{2\lambda} \epsilon^{\rho\sigma\lambda\tau}}{(k_1 + k_2)^2} = 12\frac{k_1 \cdot (k_1 + k_2) k_{2\alpha} - k_2 \cdot (k_1 + k_2) k_{1\alpha}}{(k_1 + k_2)^2} \quad (3)$$

$$v_\alpha = (k_1 - k_2)_\alpha - \frac{(k_1 + k_2)_\alpha (k_1^2 - k_2^2)}{(k_1 + k_2)^2} \quad (4)$$

The last step follows from the previous one by writing $k_1 = (k_1+k_2)/2 + (k_1-k_2)/2$ and $k_2 = (k_1+k_2)/2 - (k_1-k_2)/2$ for $k_{1\alpha}$ and $k_{2\alpha}$ respectively. On photon mass shell this proves the above identity. Off-shell it proves the one in my footnote!

2. **Decay rate for $\pi_0 \to \gamma\gamma$.**

a) Define F_{π_0} by

$$\langle 0|j^\mu_{35}(0)|\mathbf{q},\pi_0\rangle \equiv \frac{iF_{\pi_0}q^\mu}{(2\pi)^{3/2}\sqrt{2\omega_{\pi_0}(\mathbf{q})}},$$

where $j^\mu_{35} = \bar{q}\frac{\tau_3}{2}\gamma_5\gamma^\mu q$ is the 3 component of the axial isospin current. Since isospin is a very good symmetry of the strong interactions, we may approximate $F_{\pi_0} \approx F_{\pi_-} \approx 93$ MeV. Consulting the reduction formalism developed last semester, confirm the assertion in class, that the matrix element of j^μ_{35} between the vacuum and a two photon state behaves for $(k_1+k_2)^2 \to -m^2_{\pi_0}$ as

$$\langle \mathbf{k}_1\epsilon_1, \mathbf{k}_2\epsilon_2|j^\mu_{35}(0)|0\rangle \sim \frac{(k_1+k_2)^\mu}{(k_1+k_2)^2+m^2_{\pi_0}}\frac{F_{\pi_0}\mathcal{M}_{\pi_0\to\gamma\gamma}}{(2\pi)^3\sqrt{4|\mathbf{k_1}||\mathbf{k_2}|}}, \quad \text{for} \quad (k_1+k_2)^2 \to -m^2_{\pi_0}$$

where $\mathcal{M}$ is the Feynman amplitude for the decay process. (Of course, since the pion is unstable, there is not really a pole for a real value of $(k_1+k_2)^2$. In this case the reduction formula makes sense only if the time T is taken to satisfy $T << \tau_\pi$ where τ_π is the pion lifetime. In fact the pole should move into the complex plane by an amount $-im_{\pi_0}\Gamma_{\pi_0}$, with Γ_{π_0} the rate for π_0 decay in its rest frame. Below you will find that $\Gamma_{\pi_0} = O(\alpha^2 m_{\pi_0}) << m_{\pi_0}$, so the pole is extremely close to the real axis!)

Solution: The reduction formalism shows that the matrix element of $\int d^4x e^{iq\cdot x} j^\mu_{35}(x)$ has a pole at $q^2 = (k_1+k_2)^2 = -m^2$ for any particle that couples to $j^\mu_{35}|0\rangle$. Inspection of the reduction formulas shows that all of the factors in the above formula are correct.

b) As we have seen in class, the triangle anomaly implies that, in the limit of massless quarks, the matrix element of j^μ_{35} has a pole at $(k_1+k_2)^2 = 0$. We can interpret this by saying that the mass of the pion vanishes with massless up and down quarks, in which case the pole obtained in part a) just becomes the massless pole required by the anomaly. To use the anomaly to estimate π_0 decay, we must suppose that giving the up and down quarks their relatively small masses moves the pole away from zero, giving the pion its small mass, but has a very small effect on the residue of the pole. In this case, $\mathcal{M}$ can be approximated by comparison with the residue of the massless pole required by the anomaly. Read off the resulting prediction for $\mathcal{M}$ and use it to approximately calculate the rate for $\pi_0 \to \gamma\gamma$. A clear way to express the approximation is as the tree approximation to $\pi_0 \to \gamma\gamma$ using the effective Lagrangian

$$\mathcal{L}_{eff} = -\frac{1}{2}((\partial\pi_0)^2 + m^2_\pi\pi^2_0) + \frac{\alpha}{8\pi F_{\pi_0}}\pi_0\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{5}$$

where m_π is the actual mass of the π_0.

Solution: Setting $m_\pi = 0$ and taking the divergence of both sides of the result of part a), and using the anomaly, gives

$$\frac{N_c\alpha}{24\pi}\epsilon^{\mu\nu\rho\sigma}\langle \boldsymbol{k}_1\epsilon_1, \boldsymbol{k}_2\epsilon_2|F_{\mu\nu}F_{\rho\sigma}|0\rangle \approx -i\frac{F_{\pi_0}\mathcal{M}_{\pi_0\to\gamma\gamma}}{(2\pi)^3\sqrt{4|\mathbf{k_1}||\mathbf{k_2}|}}. \tag{6}$$

The matrix element on the left is the two photon vacuum matrix element of two free photon field strengths:

$$\langle \boldsymbol{k}_1\epsilon_1, \boldsymbol{k}_2\epsilon_2|F_{\mu\nu}F_{\rho\sigma}|0\rangle = (i)^2 \left((k_{1\mu}\epsilon_{1\nu} - k_{1\nu}\epsilon_{1\mu})(k_{2\rho}\epsilon_{2\sigma} - k_{1\sigma}\epsilon_{1\rho}) + (1 \leftrightarrow 2)\right) \frac{1}{(2\pi)^3\sqrt{4|\mathbf{k_1}||\mathbf{k_2}|}} \quad (7)$$

Then, putting $N_c = 3$,

$$\begin{aligned} \mathcal{M} &\approx -i\frac{\alpha}{8\pi F_\pi}\epsilon^{\mu\nu\rho\sigma}\left((k_{1\mu}\epsilon_{1\nu} - k_{1\nu}\epsilon_{1\mu})(k_{2\rho}\epsilon_{2\sigma} - k_{1\sigma}\epsilon_{1\rho}) + (1 \leftrightarrow 2)\right) \\ &= -i\frac{\alpha}{\pi F_\pi}\epsilon^{\mu\nu\rho\sigma}k_{1\mu}\epsilon_{1\nu}k_{2\rho}\epsilon_{2\sigma} \end{aligned} \quad (8)$$

In the pion rest frame $\boldsymbol{k}_2 = -\boldsymbol{k}_1$ so we can write $k_2^\mu = -k_1^\mu + 2(|\boldsymbol{k}_1|, \mathbf{0})$. Then

$$\epsilon^{\mu\nu\rho\sigma}k_{1\mu}\epsilon_{1\nu}k_{2\rho}\epsilon_{2\sigma} = -2|\boldsymbol{k}_1|\epsilon^{ijk}k_{1i}\epsilon_{1j}\epsilon_{2k} \quad (9)$$

For the total rate we require

$$\sum_{Pol}|\mathcal{M}|^2 = \frac{\alpha^2}{\pi^2F_\pi^2}8\boldsymbol{k}_1^4 \quad (10)$$

The formula for the decay rate is, in the pion rest frame

$$\Gamma = \frac{1}{2\cdot 2m_\pi}\int\frac{4\pi k_1^2 dk^1}{(2\pi)^3 4k_1^2}(2\pi)\delta(2k_1 - m_\pi)|\mathcal{M}|^2 = \frac{|\mathcal{M}|^2}{32\pi m_\pi} = \frac{\alpha^2 m_\pi^3}{64\pi^3F_\pi^2} \quad (11)$$

where identity of the two photons requires an extra factor of 1/2, and we evaluated $|\mathcal{M}|^2$ at $k_1 = m_\pi/2$.

c) Look up the experimental lifetime τ_{π_0} for the π_0. Since two photons are almost always the final state, $1/\tau_{\pi_0}$ is a good approximation to the rate for decay into $\gamma\gamma$. Compare the experimental result to the calculation of parts a) annd b), and discuss the accuracy of the approximation.

Solution: Puttinig in the numbers $m_\pi = 135$MeV, $F_\pi \approx 93$MeV, and $\alpha \approx 1/137$ gives for the rate, $\Gamma \approx 7.65 \times 10^{-6}$MeV. The lifetime is $\tau = \hbar/\Gamma \approx 8.6 \times 10^{-17}$ sec. This is to be compared to the experimental lifetime $8.52 \pm 0.18) \times 10^{-17}$, with excellent agreement! The main assumption was that the extrapolation from massless quarks to quarks with a small mass did not introduce significant errors, which seems to be confirmed by the agreement.

3. The last problem on the take home exam is important enough to justify giving it a careful second look. Rework your solution correcting any mistakes and filling in all important details. I reproduce the problem below:

From Final: In class we used a nonrenormalizable effective field theory model to obtain low energy pion scattering from spontaneously broken chiral symmetry. Repeat the calculation, using instead the linear σ model with the renormalizable Lagrangian:

$$\mathcal{L} = -\tfrac{1}{2}(\partial\phi^a)^2 - \frac{\lambda}{4}(\phi^a\phi^a - F_\pi^2)^2$$

with ϕ^a a 4-vector field under chiral $O(4)$ which is isomorphic to $SU(2)\times SU(2)$.

a) After shifting the field by its classical expectation value, write out the Lagrangian in terms of three pion fields π_a and one other scalar field σ. Show that the particle content is three massless Goldstone bosons (pions) and a massive scalar field, σ, which is the component of ϕ^a that gets an expectation value.

Solution: The minimum of the potential clearly satisfies $\phi_a\phi_a = f_\pi^2$. Solve this by picking $\phi = (f_\pi, 0, 0, 0)$. Write $\phi_4 = f_\pi + \sigma$ and call the remaining 3 components $\boldsymbol{\pi}$ a three vector. then

$$\frac{\lambda}{4}(\phi_a^2 - f_\pi^2)^2 = \frac{\lambda}{4}(\boldsymbol{\pi}^2 + \sigma^2 + 2f_\pi\sigma)^2 = \lambda f_\pi^2\sigma^2 + \lambda f_\pi\sigma(\boldsymbol{\pi}^2 + \sigma^2) + \frac{\lambda}{4}(\boldsymbol{\pi}^2 + \sigma^2)^2 \tag{12}$$

The Lagrangian in these coordinates is

$$\mathcal{L} \;=\; -\frac{1}{2}(\partial\boldsymbol{\pi})^2 - \frac{1}{2}(\partial\sigma)^2 - \lambda f_\pi^2\sigma^2 - \lambda f_\pi\sigma(\boldsymbol{\pi}^2 + \sigma^2) - \frac{\lambda}{4}(\boldsymbol{\pi}^2 + \sigma^2)^2 \tag{13}$$

This Lagrangian describes a massive σ particle with mass squared $m_\sigma^2 = 2f_\pi^2\lambda$, and three massless π particles (NGB's). The manifest symmetry is $SO(3)$, the $O(4)$ symmetry is sponteneously broken.

b) (7 points) Draw and evaluate all of the 4 point tree diagrams contributing to $\pi\pi$ scattering and to $\pi\sigma$ scattering, using the Cartesian basis for pion fields π_a, $a = 1, 2, 3$, Then changing to the charge basis $\pi_\pm = (\pi_1 \pm i\pi_2)/\sqrt{2}$, $\pi_o = \pi_3$, obtain the Feynman amplitudes for the scattering processes $\pi^+\pi^+ \to \pi^+\pi^+$, $\pi^+\pi^0 \to \pi^+\pi^0$, $\pi^+\pi^- \to \pi^+\pi^-$, $\pi^+\pi^- \to \pi^0\pi^0$, and $\pi^0\sigma \to \pi^0\sigma$.

Solution: We list the vertices from this Lagrangian:

$$\begin{aligned}
&\sigma\pi\pi:\ -2i\lambda f_\pi\delta_{ab}, \qquad \sigma\sigma\sigma: -6i\lambda f_\pi, \qquad \pi\pi\pi\pi: -2i\lambda(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\\
&\sigma\sigma\pi\pi:\ -2i\lambda\delta_{ab}, \qquad \sigma\sigma\sigma\sigma:\ -6i\lambda
\end{aligned} \tag{14}$$

For amplitudes involving 4 pions the diagrams containing two cubic vertices are, putting $\lambda = m_\sigma^2/2f_\pi^2$

$$-i\frac{(-im_\sigma^2)^2}{f_\pi^2}\left(\frac{\delta_{ab}\delta_{cd}}{m_\sigma^2 - s} + + \frac{\delta_{ac}\delta_{bd}}{m_\sigma^2 - u} + \frac{\delta_{ad}\delta_{bc}}{m_\sigma^2 - t}\right) \tag{15}$$

Adding the quartic pion vertex subtracts $1/m_\sigma^2$ from each coefficient of Kronecker deltas in parentheses, resulting in

$$\begin{aligned}
\mathcal{M}_{abcd} \;&=\; i\frac{m_\sigma^2}{f_\pi^2}\left\{\delta_{ab}\delta_{cd}\frac{s}{m_\sigma^2 - s} + \delta_{ac}\delta_{bd}\frac{u}{m_\sigma^2 - u} + \delta_{ad}\delta_{bc}\frac{t}{m_\sigma^2 - t}\right\}\\
&\to\; \frac{i}{f_\pi^2}\left\{\delta_{ab}\delta_{cd}s + \delta_{ac}\delta_{bd}u + \delta_{ad}\delta_{bc}t\right\}
\end{aligned}$$

For $\pi^+\pi^+$ put $a = b = (1 + i2)/\sqrt{2}$ and $c = d = (1 - i2)/\sqrt{2}$, for $\pi^+\pi^0$ change $b = c = 3$, for $\pi^+\pi^- \to \pi^+\pi^-$, change $b = (1 - i2)/\sqrt{2}$ and $c = (1 + i2)/\sqrt{2}$, and for $\pi^+\pi^- \to \pi^0\pi^0$ keep

$b = (1 - i2)/\sqrt{2}$ and change $b = c = d = 3$:

$$\begin{aligned}
\mathcal{M}_{\pi^+\pi^+} &= i\frac{m_\sigma^2}{f_\pi^2}\left\{\frac{u}{m_\sigma^2 - u} + \frac{t}{m_\sigma^2 - t}\right\} \to i\frac{1}{f_\pi^2}(u+t) \\
\mathcal{M}_{\pi^+\pi^0} &= i\frac{m_\sigma^2}{f_\pi^2}\left\{\frac{t}{m_\sigma^2 - t}\right\} \to i\frac{1}{f_\pi^2}t \\
\mathcal{M}_{\pi^+\pi^-\to\pi^+\pi^-} &= i\frac{m_\sigma^2}{f_\pi^2}\left\{\frac{t}{m_\sigma^2 - t} + \frac{s}{m_\sigma^2 - s}\right\} \to i\frac{1}{f_\pi^2}(s+t) \\
\mathcal{M}_{\pi^+\pi^-\to\pi^0\pi^0} &= i\frac{m_\sigma^2}{f_\pi^2}\left\{\frac{s}{m_\sigma^2 - s}\right\} \to i\frac{1}{f_\pi^2}s
\end{aligned} \tag{16}$$

For the process $\pi_o\sigma \to \pi_o\sigma$ has threes diagrams with cubic vertices and one with the quartic:

$$\mathcal{M}_{\pi^0\sigma} = i\frac{m_\sigma^4}{f_\pi^2}\left(\frac{3}{m_\sigma^2 - t} + \frac{1}{-s} + \frac{1}{-u}\right) - i\frac{m_\sigma^2}{f_\pi^2} \tag{17}$$

c) (6 points) Calculate the center of mass differential cross section for each process. Obtain the low energy behavior of the cross section in each case.

Solution: For 2 particle elastic scattering the C of M differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 s} \tag{18}$$

which applies to all of the above processes. For $\pi\pi$ scattering low energy means s, t, u all small, and ccorresponding the differential cross sections vanish at low energy. However for $\pi\sigma$ scattering $s > m_\sigma^2$ so low energy means $t = 0, s = u = m_\sigma^2$. Thus $\mathcal{M}_{\pi^0\sigma} \to 0$, and the corresponding differential cross section also vanishes at low energy by virtue of a cancellation between the different diagram contributions. This illustrates the general property that NGB's decouple at low energy universally.

d) Now add a symmetry breaking term $-c\phi^4$, an isoscalar. What does "vacuum alignment" tell you about the VEV you must use for ϕ^a? Find the mass of the pions in the presence of this symmetry breaking.

Solution: Before symmetry breaking, $\langle\phi^a\rangle = n^a f_\pi$ could be in any divection. With the symmetry breaking term $c\phi^4$ in the effective potential, the direction should be chosen to minimize $cf_\pi n^4$, which, assuming $C > 0$, happens when $n^4 = -1$ and $n^1 = n^2 = n^3 = 0$. Writing $\phi^4 = -v + \sigma$, the minimum of the potential determines v:

$$\lambda v(v^2 - f_\pi^2) = c \tag{19}$$

which can be solved perturbatively $v = f_\pi + \delta$, $\delta = c/2f_\pi + O(c^2)$. Expanding the potential about $(\sigma, \pi^a) = 0$

$$V(\sigma, \pi^a) = V(0,0) + \frac{\lambda}{2}(3v^2 - f_\pi^2)\sigma^2 + \frac{\lambda}{2}(v^2 - f_\pi^2)\boldsymbol{\pi}^2 + \text{cibic, quartic} \tag{20}$$

which shows that

$$m_\pi^2 = \lambda(v^2 - f_\pi^2) = \frac{c}{v} = \frac{c}{f_\pi} + O(c^2) \tag{21}$$

We also get a formula for the corrected

$$m_\sigma^2 = 3\lambda v^2 - \lambda f_\pi^2 = 3m_\pi^2 + 2\lambda f_\pi^2 \tag{22}$$

using which we can get $\lambda = (m_\sigma^2 - 3m_\pi^2)/(2f_\pi^2)$, which allows us to regard m_σ, m_π, f_π as the parameters of the theory (instead of λ, c, f_π. In this way we avoid the need to solve the cubic equation for v.

e) Evaluate the amplitudes for all the processes of b) in the presence of symmetry breaking. Find the threshold limit of each amplitude.

Solution: The full Lagrangian can be taken to be

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}(\partial\boldsymbol{\pi})^2 - \frac{1}{2}(\partial\sigma)^2 - \frac{m_\sigma^2}{2}\sigma^2 - \frac{m_\pi^2}{2}\boldsymbol{\pi}^2 - \frac{g}{2}\sigma(\boldsymbol{\pi}^2 + \sigma^2) - \frac{\lambda}{4}(\boldsymbol{\pi}^2 + \sigma^2)^2 \\
\lambda &= \frac{m_\sigma^2 - 3m_\pi^2}{2f_\pi^2}, \qquad g = 2\lambda v = \frac{\sqrt{(m_\sigma^2 - 3m_\pi^2)(m_\sigma^2 - m_\pi^2)}}{f_\pi}
\end{aligned} \tag{23}$$

Recalculating the amplitudes we find

$$\begin{aligned}
\mathcal{M}_{abcd} &= ig^2\left(\frac{\delta_{ab}\delta_{cd}}{m_\sigma^2 - s} + +\frac{\delta_{ac}\delta_{bd}}{m_\sigma^2 - u} + \frac{\delta_{ad}\delta_{bc}}{m_\sigma^2 - t}\right) - 2i\lambda(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \\
&= i\frac{m_\sigma^2 - 3m_\pi^2}{f_\pi^2}\left\{\delta_{ab}\delta_{cd}\frac{s - m_\pi^2}{m_\sigma^2 - s} + \delta_{ac}\delta_{bd}\frac{u - m_\pi^2}{m_\sigma^2 - u} + \delta_{ad}\delta_{bc}\frac{t - m_\pi^2}{m_\sigma^2 - t}\right\} \\
&\to i\frac{m_\sigma^2 - 3m_\pi^2}{f_\pi^2}\left\{\delta_{ab}\delta_{cd}\frac{3m_\pi^2}{m_\sigma^2 - 4m_\pi^2} + \delta_{ac}\delta_{bd}\frac{-m_\pi^2}{m_\sigma^2} + \delta_{ad}\delta_{bc}\frac{-m_\pi^2}{m_\sigma^2}\right\} \\
&\to \frac{im_\pi^2}{f_\pi^2}\{3\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\} + O(m_\pi^4)
\end{aligned}$$

The individual processes become

$$\begin{aligned}
\mathcal{M}_{\pi^+\pi^+} &= i\frac{m_\sigma^2 - 3m_\pi^2}{f_\pi^2}\left\{\frac{u - m_\pi^2}{m_\sigma^2 - u} + \frac{t - m_\pi^2}{m_\sigma^2 - t}\right\} \to -2i\frac{1 - 3m_\pi^2/m_\sigma^2}{f_\pi^2}m_\pi^2 \\
\mathcal{M}_{\pi^+\pi^0} &= i\frac{m_\sigma^2 - 3m_\pi^2}{f_\pi^2}\left\{\frac{t - m_\pi^2}{m_\sigma^2 - t}\right\} \to -i\frac{1 - 3m_\pi^2/m_\sigma^2}{f_\pi^2}m_\pi^2 \\
\mathcal{M}_{\pi^+\pi^-\to\pi^+\pi^-} &= i\frac{m_\sigma^2 - 3m_\pi^2}{f_\pi^2}\left\{\frac{t - m_\pi^2}{m_\sigma^2 - t} + \frac{s - m_\pi^2}{m_\sigma^2 - s}\right\} \to -i\frac{1 - 3m_\pi^2/m_\sigma^2}{f_\pi^2}m_\pi^2\left(1 - \frac{3}{1 - 4m_\pi^2/m_\sigma^2)}\right) \\
\mathcal{M}_{\pi^+\pi^-\to\pi^0\pi^0} &= i\frac{m_\sigma^2 - 3m_\pi^2}{f_\pi^2}\left\{\frac{s - m_\pi^2}{m_\sigma^2 - s}\right\} \to 3i\frac{1 - 3m_\pi^2/m_\sigma^2}{f_\pi^2(1 - 4m_\pi^2/m_\sigma^2)}m_\pi^2
\end{aligned} \tag{24}$$

Finally the $\pi\sigma$ amplitude becomes

$$\begin{aligned}
\mathcal{M}_{\pi^0\sigma} &= ig^2\left(\frac{3}{m_\sigma^2 - t} + \frac{1}{m_\pi^2 - s} + \frac{1}{m_\pi^2 - u}\right) - 2i\lambda \\
&= i\frac{(m_\sigma^2 - 3m_\pi^2)(m_\sigma^2 - m_\pi^2)}{f_\pi^2}\left(\frac{3}{m_\sigma^2 - t} + \frac{1}{m_\pi^2 - s} + \frac{1}{m_\pi^2 - u}\right) - i\frac{m_\sigma^2 - 3m_\pi^2}{f_\pi^2}
\end{aligned} \tag{25}$$

Threshold for this process is $s = (m_\sigma + m_\pi)^2$, $t = 0$, and $u = -(m_\sigma - m_\pi)^2$:

$$\begin{aligned}
\mathcal{M}_{\pi^0\sigma} &\rightarrow i\frac{(m_\sigma^2 - 3m_\pi^2)}{f_\pi^2}\left[(m_\sigma^2 - m_\pi^2)\left(\frac{3}{m_\sigma^2} + \frac{1}{-m_\sigma^2 - 2m_\pi m_\sigma} + \frac{1}{-m_\sigma^2 + 2m_\pi m_\sigma}\right) - 1\right] \\
&\rightarrow i\frac{(m_\sigma^2 - 3m_\pi^2)}{f_\pi^2}\left[(m_\sigma^2 - m_\pi^2)\left(\frac{3}{m_\sigma^2} - \frac{2}{m_\sigma^2 - 4m_\pi^2}\right) - 1\right] \qquad (26)
\end{aligned}$$

f) Compare your answers to those we got in class, taking into account the new physics associated with the σ particle in this model, which played no role in the class discussion (why?).

Solution: The presence of the sigma field brings in some higher energy physics than the class discussion, which assumed that energies were small *and* that m_π was significantly smaller than any other mass scale. With finite m_σ the amplitudes and cross sections have considerably more complexity. However If one neglects m_π/m_σ compared to unity, simple inspection shows that the results of this exercise go over to those we got in class. This model gives an example of how the low energy consequences of chiral symmetry breaking are universal and model independent.