Standard Model/Quantun Field Theory III

Solution Set 2

Due: Wenesday, 5 February 2020

Suggested reading: QFT Notes, Ch 25-26; Sr, Secs 75-77,83; P, Ch 19; Sc, Ch 30. Here Sr=Srednicki, P=Peskin&Schroeder, and Sc=Schwartz. These sources cover the same material, but from different poits of view. My notes are self-contained.

4. An important part of the derivation of low energy pion nucleon scattering from spontaneously broken chiral symmetry was establishing that the C terms in the effective action (25.89) in the lecturenotes gave a contribution suppressed by a factor of m_π/m_N compated to the other terms. This was sketched in class Eqs (25.99)-(25.101). Redo this calculation by first confirming that the values for the diagrams quoted in these equations follow from the effective action and then present the calculation supplying all missing steps. Be especially careful with the signs and coefficients needed for the cancellation of the order m_π terms linear in C.

Solution: Since the solution is already sketched in the notes I do not have anything else to add here. Please let me know if you need more guidance than you find in the notes.

5. Use isospin symmetry to relate pion proton scattering total cross sections to pion neutron total cross sections. Show in particular that, in the limit of exact isospin symmetry, $\sigma_{\pi^+p} - \sigma_{\pi^-p} = \sigma_{\pi^-n} - \sigma_{\pi^+n}$.

Solution: (p, n) is an isodoublet ($I = 1/2$) and (π^+, π^0, π^-) is an isotriplet ($I = 1$). Together they combine to $I = 3/2$ and $I = 1/2$, exactly parallel to angular momentum addition. Using ladder operators $T_\pm = T_1 \pm iT_2$ or consulting the analogous angular momentum Clebsch-Gordon coefficients one obtains

$$|\pi^+p\rangle = |3/2, 3/2\rangle. \quad |\pi^0p\rangle = \frac{1}{\sqrt{3}}(|3/2.1/2\rangle\sqrt{2} + |1/2, 1/2\rangle), \quad |\pi^-p\rangle = \frac{1}{\sqrt{3}}(|3/2. -1/2\rangle + |1/2, -1/2\rangle\sqrt{2})$$
$$|\pi^-n\rangle = |3/2, -3/2\rangle. \quad |\pi^0n\rangle = \frac{1}{\sqrt{3}}(|3/2. -1/2\rangle\sqrt{2} - |1/2, -1/2\rangle), \quad |\pi^+n\rangle = \frac{1}{\sqrt{3}}(|3/2.1/2\rangle - |1/2, 1/2\rangle\sqrt{2})$$

To relate these results to total cross sections we use the optical theorem to identify the total cross section with the imaginary part of the forward elastic amplitude. By isospin invariance there are two independent scattering amplitudes for isospin 3/2 and 1/2: call their respective total cross sections $\sigma_{3/2}, \sigma_{1/2}$. Then

$$\sigma_{\pi^+p} = \sigma_{3/2}, \quad \sigma_{\pi^-p} = \frac{1}{3}\sigma_{3/2} + \frac{2}{3}\sigma_{1/2}, \qquad \sigma_{\pi^0p} = \frac{2}{3}\sigma_{3/2} + \frac{1}{3}\sigma_{1/2}$$
$$\sigma_{\pi^-n} = \sigma_{3/2}, \quad \sigma_{\pi^+n} = \frac{1}{3}\sigma_{3/2} + \frac{2}{3}\sigma_{1/2}, \qquad \sigma_{\pi^0n} = \frac{2}{3}\sigma_{3/2} + \frac{1}{3}\sigma_{1/2}$$

Comparing the two lines we see that $\sigma_{\pi^\pm p} = \sigma_{\pi^\mp n}$ and $\sigma_{\pi^0p} = \sigma_{\pi^0n}$. So the desired relation $\sigma_{\pi^+p} - \sigma_{\pi^-p} = \sigma_{\pi^-n} - \sigma_{\pi^+n}$ follows. This confirms that the two independnet formulas for g_A from the Adler-Weisberger sum rule give the same answer.

6. Anomalies and Instantons

a) For nonabelian auge theory, calculate $\partial_\mu K^\mu$ where

$$K^\mu = \frac{g_3^2 n_F}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left[A_\nu\partial_\rho A_\sigma - \frac{2ig_3}{3}A_\nu A_\rho A_\sigma\right] \tag{1}$$

where $A_\mu = \sum_a t_a A_\mu^a$ is the matrix represntaion of the gauge field, and show that $\partial_\mu K^\mu$ matches the chiral anomaly.

Solution:

$$\begin{aligned}\partial_\mu K^\mu &= \frac{g_3^2 n_F}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left[\partial_\mu A_\nu\partial_\rho A_\sigma - \frac{2ig_3}{3}[\partial_\mu A_\nu A_\rho A_\sigma + \partial_\mu A_\rho A_\sigma A_\nu + \partial_\mu A_\sigma A_\nu A_\rho]\right]\\ &= \frac{g_3^2 n_F}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left[\partial_\mu A_\nu\partial_\rho A_\sigma - 2ig_3\partial_\mu A_\nu A_\rho A_\sigma\right]\end{aligned} \tag{2}$$

where the last line follows from renaming dummy indices and permuting the indies of the epsilon symbol. On the other hand

$$\begin{aligned}\epsilon^{\mu\nu\rho\sigma}TrF_{\mu\nu}F_{\rho\sigma} &= 4\epsilon^{\mu\nu\rho\sigma}Tr(\partial_\mu A_\nu - igA_\mu A_\nu)(\partial_\rho A_\sigma - igA_\rho A_\sigma)\\ &= 4\epsilon^{\mu\nu\rho\sigma}Tr[\partial_\mu A_\nu\partial_\rho A_\sigma - 2ig\partial_\mu A_\nu A_\rho A_\sigma - g^2 A_\mu A_\nu A_\rho A_\sigma]\end{aligned} \tag{3}$$

But $\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}A_\mu A_\nu A_\rho A_\sigma = 0$ because the trace is cyclically invariant but the epsilon symbol is anti-cyclic. Thus

$$\partial_\mu K^\mu = \frac{g_3^2 n_F}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}F_{\mu\nu}F_{\rho\sigma} \tag{4}$$

which is exactly the strength of the anomaly.

b) Compute the integral

$$\int d^3\theta\epsilon^{ijk}\mathrm{Tr}\left[\Omega^\dagger\partial_i\Omega\Omega^\dagger\partial_j\Omega\Omega^\dagger\partial_k\Omega\right] = \pm 12\pi^2 \tag{5}$$

where $\Omega(\boldsymbol{\theta}) = \pm\sqrt{1-\boldsymbol{\theta}^2}I + i\boldsymbol{\theta}\cdot\boldsymbol{\tau}$ maps S_3 to $SU(2)$.

Solution:

$$\begin{aligned}\partial_k\Omega &= \mp\frac{\theta_k}{\sqrt{1-\boldsymbol{\theta}^2}}I + i\tau_k\\ \Omega^{-1}\partial_k\Omega &= (\pm\sqrt{1-\boldsymbol{\theta}^2}I - i\boldsymbol{\theta}\cdot\boldsymbol{\tau})\partial_k\Omega = -\theta_k I \pm i\sqrt{1-\boldsymbol{\theta}^2}\tau_k \pm i\frac{\theta_k\boldsymbol{\theta}\cdot\boldsymbol{\tau}}{\sqrt{1-\boldsymbol{\theta}^2}} + \boldsymbol{\theta}\cdot\boldsymbol{\tau}\tau_k\\ &= \pm i\sqrt{1-\boldsymbol{\theta}^2}\tau_k \pm i\frac{\theta_k\boldsymbol{\theta}\cdot\boldsymbol{\tau}}{\sqrt{1-\boldsymbol{\theta}^2}} + i\epsilon_{jkl}\theta_j\tau_l \equiv A_{kl}\tau_l\end{aligned} \tag{6}$$

It will help in the trace evaluation to express

$$\begin{aligned}
\Omega^\dagger\partial_j\Omega\Omega^\dagger\partial_k\Omega &= -\partial_j\Omega^\dagger\Omega\Omega^\dagger\partial_k\Omega = -\partial_j\Omega^\dagger\partial_k\Omega \\
&= -\left(\mp\frac{\theta_j}{\sqrt{1-\boldsymbol{\theta}^2}}I - i\tau_j\right)\left(\mp\frac{\theta_k}{\sqrt{1-\boldsymbol{\theta}^2}}I + i\tau_k\right) \\
&= -\frac{\theta_j\theta_k}{1-\boldsymbol{\theta}^2}I \mp i\frac{\theta_k\tau_j - \theta_j\tau_k}{\sqrt{1-\boldsymbol{\theta}^2}} - \delta_{jk} - i\epsilon^{jkl}\tau_l \\
\epsilon^{ijk}\Omega^\dagger\partial_j\Omega\Omega^\dagger\partial_k\Omega &= \pm 2i\epsilon^{ijk}\frac{\theta_j\tau_k}{\sqrt{1-\boldsymbol{\theta}^2}} - i\epsilon^{ijk}\epsilon^{jkl}\tau_l = \pm 2i\epsilon^{ijk}\frac{\theta_j\tau_k}{\sqrt{1-\boldsymbol{\theta}^2}} - 2i\tau_i \equiv B_{ik}\tau_k \quad (7)
\end{aligned}$$

Then

$$\begin{aligned}
\epsilon^{ijk}\mathrm{Tr}\left[\Omega^\dagger\partial_i\Omega\Omega^\dagger\partial_j\Omega\Omega^\dagger\partial_k\Omega\right] &= A_{il}B_{ik}\mathrm{Tr}\tau_l\tau_k = 2A_{il}B_{il} \\
&= -4i\mathrm{Tr}A \pm 4i\frac{\epsilon^{ijl}\theta_j}{\sqrt{1-\boldsymbol{\theta}^2}}A_{il} \quad (8)
\end{aligned}$$

Now

$$\begin{aligned}
A_{kl} &= \pm i\sqrt{1-\boldsymbol{\theta}^2}\delta_{kl} \pm i\frac{\theta_k\boldsymbol{\theta}_l}{\sqrt{1-\boldsymbol{\theta}^2}} + i\epsilon_{jkl}\theta_j \\
4i\mathrm{Tr}A &= \mp 4\frac{3-2\boldsymbol{\theta}^2}{\sqrt{1-\boldsymbol{\theta}^2}}, \qquad \mp 4i\epsilon^{ijl}\theta_j A_{il} = \pm 4\epsilon^{ijl}\theta_j\epsilon^{kil}\theta_k = \mp 8\boldsymbol{\theta}^2 \\
\int d^3\theta\epsilon^{ijk}\mathrm{Tr}\left[\Omega^\dagger\partial_i\Omega\Omega^\dagger\partial_j\Omega\Omega^\dagger\partial_k\Omega\right] &= \pm 48\pi\int_0^1\frac{\theta^2 d\theta}{\sqrt{1-\boldsymbol{\theta}^2}} = \pm 12\pi^2 \quad (9)
\end{aligned}$$

as desired.

c) Calculate the field strengths for the one instanton potentials in an $SU(2)$ gauge theory:

$$A_\mu = \frac{i}{g}\begin{cases}\dfrac{-i\boldsymbol{x}\cdot\boldsymbol{\tau}}{r^2+R^2} & \mu = 4 \\ \dfrac{ix^4\tau^k + i\boldsymbol{\tau}\times\boldsymbol{x})^k}{r^2+R^2} & \mu = k\end{cases} \quad (10)$$

and prove that A_μ solve the classical (Euclidean) field equations by showing that the field strengths are self- or anti- dual.

Solution: We calculate separately F_{kl} and F_{4k}.

$$\begin{aligned}
F_{kl} &= \partial_k A_l - \partial_l A_k - ig[A_k, A_l] \\
-ig\partial_k A_l &= \frac{+i\epsilon^{lmk}\tau_m}{r^2+R^2} - 2x^k\frac{ix^4\tau^l + i(\boldsymbol{\tau}\times\boldsymbol{x})^l}{(r^2+R^2)^2} \\
[-igA_k, -igA_l] &= A_{km}A_{ln}[\tau^m,\tau^n] = 2iA_{km}A_{ln}\epsilon^{mnp}\tau\ au^p \\
&= \frac{(x^4)^2\epsilon^{klr} + x^4(-\delta_{lr}x^k + \delta_{kr}x^l) + \epsilon^{klp}x^p x^r}{(r^2+R^2)^2}\tau^r \\
-igF_{kl} &= \frac{2i\epsilon^{lmk}\tau_m R^2}{(r^2+R^2)^2}
\end{aligned}$$

$$
\begin{aligned}
-ig\partial_4 A_k &= \frac{i\tau^k}{r^2+R^2} - 2x^4 \frac{ix^4\tau^k + i(\boldsymbol{\tau}\times\boldsymbol{x})^k}{(r^2+R^2)^2} \\
-ig\partial_4 A_k &= \frac{i\tau^k}{r^2+R^2} - 2x^4 \frac{ix^4\tau^k + i(\boldsymbol{\tau}\times\boldsymbol{x})^k}{(r^2+R^2)^2} \\
-ig\partial_k A_4 &= \frac{-i\boldsymbol{\tau}_k}{r^2+R^2} - 2x_k \frac{-i\boldsymbol{x}\cdot\boldsymbol{\tau}}{(r^2+R^2)^2} \\
[-igA_4, -igA_k] &= 2\frac{ix^4(\boldsymbol{\tau}\times\boldsymbol{x})^k + i((\boldsymbol{\tau}\times\boldsymbol{x})\times\beta\ fsx)^k}{(r^2+R^2)^2} \\
&= 2\frac{ix^4(\boldsymbol{\tau}\times\boldsymbol{x})^k + i(\boldsymbol{\tau}\cdot\boldsymbol{x}x^k - \boldsymbol{x}^2\tau^k)}{(r^2+R^2)^2} \\
-igF_{4k} &= -ig\partial_4 A_k + ig\partial_k A_4 + (-ig)^2[A_4, A_k] = \frac{2i\tau_k R^2}{(r^2+R^2)^2} \qquad (11)
\end{aligned}
$$

We see that $F_{kl} = \epsilon^{klm} F_{4m}$ which is the statement of self duality. Hence the configuration is a minimum of the classical action and hence a solution of the classical equations of motion.

7. Consider the generalization of the winding number formula to general dimension d.

a) Show that

$$
\partial_{\mu_1}\epsilon^{\mu_1\cdot\mu_d}\mathrm{Tr}U^\dagger\partial_{\mu_2}U\cdots U^\dagger\partial_{\mu_d}U = \begin{cases} 0 & d \quad \text{odd} \\ -\epsilon^{\mu_1\cdot\mu_d}\mathrm{Tr}U^\dagger\partial_{\mu_1}U\cdots U^\dagger\partial_{\mu_d}U & d \quad \text{even} \end{cases} \qquad (12)
$$

Solution: The derivative only contributes when it hits an $\Omega^\dagger$. When d is odd there will be an even number of such contributions and they alternate in sign, cancelling in pairs. When d is even there are an odd number of contribu tions again alternating in sign so the cancellation in pairs leaves one behind.

$$
\partial_\mu\omega^\dagger = -\Omega^\dagger\partial_\mu\Omega\Omega^\dagger \qquad (13)
$$

so the left over contribution is just what is shown in the quoted result.

b) When d is odd, the quantity

$$
\omega = \epsilon^{\mu_1\cdot\mu_d}\mathrm{Tr}U^\dagger\partial_{\mu_1}U\cdots U^\dagger\partial_{\mu_d}U \qquad (14)
$$

is not a total derivative. Show, however, that its variation under $U \to U + \delta U$ to first order in δU is a total derivative. If you wish you may specialize to the special case $d = 5$ which figures in our discussion of the WZW term.

Solution: First calculate

$$
\delta(U^\dagger\partial U = -U^\dagger\delta U U^\dagger\partial U + U^\dagger\partial\delta U = \partial(U^\dagger\delta U) - U^\dagger\delta U U^\dagger\partial U + U^\dagger\partial U U^\dagger\delta U \qquad (15)
$$

When inserted in the trace, the cyclic prpoerty of the trace and that of the odd index epsilon symbol show that the contributions of the last two terms cancel. Thus we can assert that

$$
\delta\omega = \partial_{\mu_1}(\epsilon^{\mu_1\cdots\mu_d}\mathrm{Tr}U^\dagger\delta U U^\dagger\partial_{\mu_2}U\cdots U^\dagger\partial_{\mu_d}U) \qquad (16)
$$

as desired.