Standard Model/Quantun Field Theory III

Solution Set 3

Due: Wednesday, 19 February 2020

Suggested reading: QFT Notes, Ch 27; Sr, Secs 93-94; P, Ch 19; Sc, Ch 28-30. Here Sr=Srednicki, P=Peskin&Schroeder, and Sc=Schwartz.

8. In this problem we address how to include the θ parameter in the chiral effective Lagrangian for pions and nucleons and then identify diagrams that give an electric dipole moment to the neutron. In class we have learned that it is actually $\bar\theta = \theta + n_F\alpha$, where α is a phase in the mass matrix M, that is measurable: if M has a zero eigenvalue, physics is independent of θ. If M is brought to diagonal form by $SU(n_F)$ matrices, the diagonal entries can all be made positive times a common phase that can be identified with $\bar\theta$. Then $M = me^{i\bar\theta/n_F}$ where m is diagonal with positive entries.

a) Consider first the alignment problem: Find the minimum U_0 of the chiral symmetry breaking terms

$$-\text{Tr}\ m(Ue^{i\bar\theta/n_F} + U^\dagger e^{-i\bar\theta/n_F}), \qquad U \in SU(n_F)$$

for $n_F = 2$. Note that it is the restriction of the order parameter U to have unit determinant that prevents $\bar\theta$ from being set to zero: the $U(1)$ problem is solved in the effective action by banishing the associated NGB.

Solution: Since we have taken m diagonal, we may take the minimizing U to also be diagonal with diagonal entries $e^{i\alpha}$ and $e^{-i\alpha}$ so U will have determinant unity. Then we minimize

$$\begin{aligned} -\text{Re}(m_u e^{i(\alpha+\bar\theta/2)} + m_d e^{i(-\alpha+\bar\theta/2)}) &= -(m_u\cos(\alpha+\bar\theta/2) + m_d\cos(\alpha-\bar\theta/2) \\ &= -(m_u+m_d)\cos\alpha\cos\frac{\bar\theta}{2} + (m_u - m_d)\sin\alpha\sin\frac{\bar\theta}{2} \end{aligned}$$

Setting its derivative w.r.t. α to zero gives

$$\tan\alpha = \frac{m_d - m_u}{m_d + m_u}\tan\frac{\bar\theta}{2} \qquad (1)$$

which determines U_0. Assuming $\bar\theta$ is very small allows the approximation

$$\begin{aligned} \alpha &\approx \frac{m_d - m_u}{m_d + m_u}\frac{\bar\theta}{2} \\ \alpha + \frac{\bar\theta}{2} &\approx \frac{m_d}{m_d + m_u}\bar\theta \\ \alpha - \frac{\bar\theta}{2} &\approx -\frac{m_u}{m_d + m_u}\bar\theta \end{aligned}$$

leading to

$$mU_0 e^{i\bar\theta/2} \approx m + i\bar\theta\frac{m_u m_d}{m_d + m_u} \qquad (2)$$

b) The NGB's (i.e. pions) can then be described by writing $U(x) = U_0 e^{i\pi^a \tau_a/F}$, and constructing the effective action as in Eq(25.89). To allow for the proton neutron mass difference requires additional terms bilinear in the nucleon fields with coefficients linear in M (see, for example, Srednicki, Sec 94). Expanding such an effective Lagrangian up to terms linear in π_a leads to Yukawa interaction terms

$$-i\frac{g_A m_N}{F}\pi^a \bar{N}\tau_a\gamma_5 N - \frac{c\mu\bar{\theta}}{F}\pi^a\bar{N}\tau_a N$$

where c is a number of order unity which connects quark mass differences to the neutron proton mass difference and $\mu \approx m_u m_d/(m_u + m_d)$. Assuming these vertices, draw the one loop $N\bar{N}\gamma$ diagrams describing the emission of a photon from a virtual charged pion line connecting a πnp vertex from the first term to another one from the second term. Write down the amplitude for each diagram as an integral over loop momentum.

Solution: The vertex from the first term is $(g_A m_N/F)\tau_a\gamma_5$ and from the second term is $-i(c\mu\bar{\theta}/F)\tau_a$. We want the pion to be charged, so we write

$$\pi_1\tau_1 + \pi_2\tau_2 = (\pi_1 + i\pi_2)(\tau_1 - i\tau_2)/2 + (\pi_1 - i\pi_2)(\tau_1 + i\tau_2)/2 = \sqrt{2}\pi_+ t_- + \sqrt{2}\pi^- t_- + \quad (3)$$

where $t_\pm$ are isospin raising and lowering operators acting on the nucleon doublet, i.e. $t_+|n\rangle = |p\rangle$ and $t_-|p\rangle = |n\rangle$. Thus the two em 1 loop vertex diagrams sum to

$$\left(\sqrt{2}\frac{g_A m_N}{F}\right)\left(-i\sqrt{2}\frac{c\mu\bar{\theta}}{F}ie\right)(-i)^3\int\frac{d^4l}{(2\pi)^4}\frac{\bar{u}_n'(\gamma_5(m_N - \gamma\cdot(p-l)) + (m_N - \gamma\cdot(p-l)\gamma_5)u_n}{(m_N^2 + (p-l)^2)(m_\pi^2 + l^2)(m_\pi^2 + (l+q)^2)\gamma_5}(q+2l)^\mu$$

$$= ie\left(2\frac{c\mu\bar{\theta}g_A m_N}{F^2}\right)\int\frac{d^4l}{(2\pi)^4}\frac{\bar{u}_n'\gamma_5 2m_N u_n(q+2l)^\mu}{(m_N^2 + (p-l)^2)(m_\pi^2 + l^2)(m_\pi^2 + (l+q)^2}$$

c) Explain why the diagrams of part b) predict an electric dipole moment for the neutron. It is meaningful even though derived from a chiral effective action, because it has an IR divergence if $m_\pi \to 0$. Cutting off the UV at $p = F$ calculate the coefficient of $\ln(m_\pi/F)$ in these diagrams. (This "chiral log" comes from the low momentum part of the loop integral.)

Solution: The reason the result of part b) predicts an electric dipole moment is contained in behavior of the the factor $\bar{u}'\gamma_5 u$ as $q \equiv p' - p$ goes to zero. To display this behavior, consider the matrix element

$$\begin{aligned}\bar{u}'\gamma_5[q\cdot\gamma, \gamma^\mu]u &= \bar{u}'\gamma_5((m_N - p\cdot\gamma)\gamma^\mu - \gamma^\mu(p'\cdot\gamma + m_N))u \\ &= \bar{u}'\gamma_5(\gamma^\mu(p\cdot\gamma) + 2p^\mu + (p'\cdot\gamma)\gamma^\mu + 2p'^\mu)u \\ &= 2(p+p')^\mu\bar{u}'\gamma_5 u \quad (4)\end{aligned}$$

Taking $q \to 0$ (so $p' \to p$) this equation implies that

$$p^\mu\bar{u}'\gamma_5 u = \frac{1}{4}\bar{u}'\gamma_5[q\cdot\gamma, \gamma^\mu]u + O(q^2) = -\frac{i}{2}\bar{u}'\gamma_5 q_\nu\sigma^{\nu\mu}u + O(q^2) \quad (5)$$

our experience with QED tells us that the matrix element $\bar{u}'\sigma^{\nu\mu}u$ is proportional to the magnetic moment of the neutron described by the coupling $F_{\mu\nu}\sigma^{\mu\nu}$. But the identity $-2i\gamma_5\sigma^{\mu\nu} =$

$\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}$ tells us to change the coupling to $F_{\mu\nu}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = 2\tilde{F}^{\rho\sigma}\sigma_{\rho\sigma}$. The dual field strength $\tilde{F}$ just interchanges the role of electric and magnetic fields in F. Thus the extra γ_5 tells that the matrix element is proportional to the electric dipole moment. The factor of p^μ is produced because at $q = 0$ the loop integral must be proportional to p^μ. To get the chiral log contributing to the electric dipole moment, can set $q = 0$ in the loop momentum integral:

$$\int \frac{d^4l}{(2\pi)^4} \frac{(2l)^\mu}{(m_N^2 + (p-l)^2)(m_\pi^2 + l^2)^2} \equiv A p^\mu$$

since, with $q = 0$, the integral over l must be proportional to p^μ. To evaluate A dot both sides into p to get

$$A p^2 \quad = \quad \int \frac{d^4l}{(2\pi)^4} \frac{(2l \cdot p)}{(2l \cdot p - l^2)(m_\pi^2 + l^2)^2}$$

To capture the infrared, cut off the l integral at a value much less that p, and drop the l^2 compared to $l \cdot p$

$$\begin{aligned} -A m_N^2 \quad &\sim \quad \frac{i}{8\pi^2} \int_0^\Lambda l^3 dl \frac{1}{(l^2 + m_\pi^2)^2} = \frac{1}{16\pi^2} \int_0^{\Lambda^2} \frac{u du}{(u + m_\pi^2)^2} \\ &\sim \quad \frac{i}{16\pi^2} \left[\ln \frac{\Lambda^2}{m_\pi^2} - 1 + O(m_\pi^2/\Lambda^2) \right] \end{aligned} \tag{6}$$

displaying the chiral log, The factor of i comes from the Wick rotation.

9. We shall soon need the generic electroweak vacuum polarization fermionic contributions in calculating the self-energies of the electroweak vector bosons, which determine the contribution of the top quark to an important parameter ρ of the standard model. The two vertices are $\gamma^\mu(1+h_2\gamma_5)$ and $\gamma^\nu(1 + h_2\gamma_5)$ and the masses of the two fermion propagators are different m_1, m_2. Start with the expression

$$\Pi_{\mu\nu}(k) = \int_0^1 dx \int \frac{d^D p_E}{(2\pi)^4} \frac{\mathrm{Tr}\gamma_\mu(1 + h_1\gamma_5)(m_1 - \gamma \cdot (p - k(1-x)))\gamma_\nu(1 + h_2\gamma_5)(m_2 - \gamma \cdot (p + kx))}{(m_1^2 x + m_2^2(1-x) + p^2 + k^2 x(1-x))^2},$$

obtained after the Feynman trick to combine denominators and the shift of $p \to p + xk$, and also after the Wick rotation. For example the usual QED vacuum polarization is the case $m_1 = m_2 = m_e$ and $h_1 = h_2 = 0$.

a) Calculate the trace of Dirac gamma matrices in the numerator and show that after averaging over directions of p^μ, the numerator becomes (in D dimensions)

$$2^{D/2} \left(-m_1 m_2 (1 - h_1 h_2)\eta_{\mu\nu} + (1 + h_1 h_2) \left[\left(\frac{2}{D} - 1 \right) p^2 \eta_{\mu\nu} - x(1-x)(2k_\mu k_\nu - k^2 \eta_{\mu\nu}) \right] \right) \tag{7}$$

Solution: The terms linear in p will average to 0, and then the terms linear in γ_5 will vanish for lack of 4 independent directions. The p independent terms are

$$\begin{aligned} &m_1 m_2 (1 - h_1 h_2) \mathrm{Tr}\gamma_\mu\gamma_\nu - x(1-x)(1 + h_1 h_2)\mathrm{Tr}\gamma_\mu k \cdot \gamma \gamma_\nu k \cdot \gamma \\ &\quad = 2^{D/2} \left(-m_1 m_2 (1 - h_1 h_2)\eta_{\mu\nu} - x(1-x)(1 + h_1 h_2)(2k_\mu k_\nu - k^2 \eta_{\mu\nu}) \right) \end{aligned} \tag{8}$$

Finally, the terms quadratic in p are

$$\begin{aligned}(1+h_1h_2)\mathrm{Tr}\gamma_\mu\gamma\cdot p\gamma_\nu\gamma\cdot p &= 2^{D/2}(1+h_1h_2)(2p_\mu p_\nu - p^2\eta_{\mu\nu}) \\ &\to 2^{D/2}(1+h_1h_2)\left(\frac{2}{D}-1\right)p^2\eta_{\mu\nu} \end{aligned} \qquad (9)$$

where the last step averages over angles in D dimensional spacetime. Putting everything together gives the desired result.

b) Derive the following identity which enables the evaluation of the integral over loop momentum:

$$\int \frac{d^Dp}{(2\pi)^D}\frac{(p^2)^m}{(p^2+A^2)^n} = \frac{(A^2)^{D/2+m-n}}{(4\pi)^{D/2}}\frac{\Gamma(m+D/2)\Gamma(n-m-D/2)}{\Gamma(D/2)\Gamma(n)}.$$

Since the integrand depends only on the length of p, you may use "polar" coordinates in D dimensions:

$$\int d^Dp \to \Omega_D\int_0^\infty p^{D-1}dp.$$

Where Ω_D is the result of integrating over angles. To find its value integrate the function $e^{-p\cdot p}$ in Cartesian coordinates and polar coordinates and compare. The remaining "radial" integral over p can be transformed into a standard representation of the Euler beta function $B(x,y) \equiv \Gamma(x)\Gamma(y)/\Gamma(x+y) = \int_0^1 dt t^{x-1}(1-t)^{y-1}$. Specialize to the cases $n=2$ and $m=0,1$.

Solution: Call the desired integral I. Then

$$\begin{aligned} I &= \frac{\Omega_D}{(2\pi)^D}\int_0^\infty p^{D-1}dp\frac{(p^2)^m}{(p^2+A^2)^n} = \frac{\Omega_D A^{D+2(m-n)}}{2(2\pi)^D}\int_0^\infty du\frac{u^{m+(D-2)/2}}{(u+1)^n} \\ &= \frac{\Omega_D A^{D+2(m-n)}}{2(2\pi)^D}\int_1^\infty du u^{-n}(u-1)^{m+(D-2)/2} \\ &= \frac{\Omega_D A^{D+2(m-n)}}{2(2\pi)^D}\int_0^1 dv v^{n-1-m-D/2}(1-v)^{m+(D-2)/2} \\ &= \frac{\Omega_D A^{D+2(m-n)}}{2(2\pi)^D}\frac{\Gamma(n-m-D/2)\Gamma(m+D/2)}{\Gamma(n)} \end{aligned} \qquad (10)$$

To get Ω_D

$$\begin{aligned} \pi^{D/2} &= \int d^Dp e^{-p\cdot p} = \Omega_D\int_0^\infty p^{D-1}dp e^{-p^2} = \frac{\Omega_D}{2}\int_0^\infty u^{(D-2)/2}dp e^{-u} = \frac{\Omega_D}{2}\Gamma(D/2) \\ \Omega_D &= \frac{2\pi^{D/2}}{\Gamma(D/2)} \end{aligned} \qquad (11)$$

Putting everything together gives the desired formula.

c) Evaluate $\Pi_{\mu\nu}(k)$ using dimensional regularization. The answer is $\Pi_{\mu\nu}(k) = \eta_{\mu\nu}\Pi_0(k^2) - (k_\mu k_\nu - k^2\eta_{\mu\nu})\Pi(k^2)$ where

$$\begin{aligned} \Pi(k^2) &= \frac{1+h_1h_2}{2\pi^2}\int_0^1 dx x(1-x)\ln\left\{\frac{\Lambda^2}{A^2(x)}\right\} \\ \Pi_0(k^2) &= \frac{1}{4\pi^2}\int_0^1 dx\left((1+h_1h_2)[m_1^2x+m_2^2(1-x)] - m_1m_2(1-h_1h_2)\right)\ln\left\{\frac{\Lambda^2}{A^2(x)}\right\} \end{aligned}$$

with $A^2 = m_1^2 x + m_2^2(1-x) + k^2 x(1-x)$, and the divergence as $d \to 4$ has been combined with the μ parameter that defines the scale in dimensional regularization to give a cutoff parameter Λ^2.

Solution:We apply the formula of part b) for $n = 2$ and $m = 0, 1$, and $A^2 = x(1-x)k^2 + m_1^2 x + m_2(1-x)$.

$$I_0 = \frac{(A^2)^{D/2-2}}{(4\pi)^{D/2}} \frac{\Gamma(2-D/2)}{\Gamma(2)}, \qquad I_1 = \frac{(A^2)^{D/2-1}}{(4\pi)^{D/2}} \frac{D\Gamma(1-D/2)}{2\Gamma(2)} = A^2 \frac{D/2}{1-D/2} I_0 \quad (12)$$

Then the x integrand becomes

$$2^{D/2} I_0 \left(-m_1 m_2 (1-h_1 h_2)\eta_{\mu\nu} + (1+h_1 h_2)\left[A^2 \eta_{\mu\nu} - x(1-x)(2k_\mu k_\nu - k^2 \eta_{\mu\nu})\right]\right) \quad (13)$$

so we read off

$$\begin{aligned} \Pi(k^2) &= (1+h_1 h_2) \int_0^1 dx x(1-x) \left[\frac{2}{(2\pi)^{D/2}} \frac{\Gamma(2-D/2)}{\Gamma(2)} - \frac{1}{2\pi^2} \ln A^2\right] \\ &\equiv (1+h_1 h_2) \int_0^1 dx x(1-x) \frac{1}{2\pi^2} \ln \frac{\Lambda^2}{A^2} \end{aligned} \quad (14)$$

Then the remaining x integrand becomes

$$2^{D/2} I_0 \left(-m_1 m_2 (1-h_1 h_2)\eta_{\mu\nu} + (1+h_1 h_2)\left[(m_1^2 x + m_2^2(1-x)\eta_{\mu\nu}\right]\right) \quad (15)$$

and the result for Π_0 follows.