Standard Model/Quantun Field Theory III

Solution Set 4

Due: Wednesday, 11 March 2020

Suggested reading: QFT Notes, Ch 28; Sr, Secs 88-90; P, Ch 21; Sc, Ch 31. Here Sr=Srednicki, P=Peskin&Schroeder, and Sc=Schwartz.

10. τ **decay**. Since the τ lepton is so massive, it can decay into many more final states than the μ could. But the options for decay into leptons is, at lowest order, limited to the two processes $\tau \to \mu + \nu_\tau + \bar{\nu}_\mu$ and $\tau \to e + \nu_\tau + \bar{\nu}_e$. Adapt our result for the decay rate of the μ to obtain the decay rates for these two processes. Look up the lifetime and branching fractions for τ decay and compare your lowest order results to the data. How good is the approximation of setting $m_\mu = m_e = 0$?

Solution: Setting $m_e = m_\mu = 0$, the rate for τ beta decay to μ is the same as to e, and given by $\Gamma = m_\tau^5 G_F^2/(192\pi^3)$. Under this assumption the rate is enhanced over the beta decay rate of the muon by a factor $(m_\tau/m_\mu)^5 = (1776.86/105.658)^5 \approx 1.34 \times 10^6$. The experimental branching fractions into μ, e are 0.1739 and 0.1782 respectively which agrees within 2% with the massless assumption. The ratio of the rate for τ beta decay to μ beta decay is

$$\frac{\Gamma_{\tau\to l\nu_\tau\bar{\nu}_l}}{\Gamma_{\mu\to e\nu_\mu\bar{\nu}_e}} = \frac{\tau_\mu}{\tau_\tau}(\text{B.F.})_l = \frac{2.197\times 10^{-6}}{2.903\times 10^{-13}} \times (.174, .178) = (1.31, 1.35)\times 10^6 \tag{1}$$

which compares very well with the $m_\mu = m_e = 0$ approximation.

11. **Photon Z-Boson Interference**

a) Calculate the differential cross section for the reaction

$$e^+ + e^- \to \mu^+ + \mu^-$$

in the center of mass frame and for energies small compared to M_Z but large compared to m_μ. Include contributions from both the photon and the Z boson. In this kinematic region the Z boson term is small compared to the photon term, so in the squared amplitude you may drop the square of the Z term. For simplicity, assume unpolarized e's and unobserved spin of the μ's

Solution: Let the e^-, e^+ momenta be p, q and the corresponding μ momenta be p', q'. The Feynman amplitudes for the two diagrams are

$$\begin{aligned}\mathcal{M}_\gamma &= \frac{ie^2}{(p+q)^2}\bar{u}(p')\gamma^\lambda v(q')\bar{v}(q)\gamma_\lambda u(p)\\ \mathcal{M}_Z &= \frac{ig_2^2}{16M_Z^2\cos^2\theta_W}\bar{u}(p')\gamma^\lambda[1-\gamma_5-4\sin^2\theta_W]v(q')\bar{v}(q)\gamma_\lambda[1-\gamma_5-4\sin^2\theta_W]u(p)\end{aligned} \tag{2}$$

We approximate

$$|\mathcal{M}_\gamma + \mathcal{M}_Z|^2 \approx |\mathcal{M}_\gamma|^2 + \mathcal{M}_\gamma^*\mathcal{M}_Z + \mathcal{M}_\gamma\mathcal{M}_Z^*$$

and we neglect the lepton masses. We also average over initial spins and sum over final spins. Taking over a past QED calculation

$$
\begin{aligned}
\langle|\mathcal{M}_\gamma|^2\rangle &\approx \frac{8e^4}{(p+q)^4}[q\cdot p'q'\cdot p + q\cdot q'p\cdot p'] \\
\langle\mathcal{M}_\gamma\mathcal{M}_Z^*\rangle &= \frac{1}{4}\frac{e^2g_2^2}{16M_Z^2(p+q)^2\cos^2\theta_W}\mathrm{Tr}\gamma\cdot p'\gamma^\lambda\gamma\cdot q'\gamma^\kappa[1-\gamma_5-4\sin^2\theta_W] \\
&\quad \times\mathrm{Tr}\gamma\cdot q\gamma_\lambda\gamma\cdot p\gamma_\kappa[1-\gamma_5-4\sin^2\theta_W] \\
&= 4\frac{e^2g_2^2}{16M_Z^2(p+q)^2\cos^2\theta_W}\Big[[1-4\sin^2\theta_W](p'^\lambda q'^\kappa + q'^\lambda p'^\kappa - p'\cdot q'\eta^{\kappa\lambda}) + i\epsilon^{\mu\lambda\nu\kappa}p'_\mu q'_\nu] \\
&\quad \times[1-4\sin^2\theta_W](p_\lambda q_\kappa + q_\lambda p_\kappa - p\cdot q\eta_{\kappa\lambda}) + i\epsilon_{\rho\lambda\sigma\kappa}p^\rho q^\sigma]\Big] \\
&= 8\frac{e^2g_2^2}{16M_Z^2(p+q)^2\cos^2\theta_W} \\
&\quad \Big[(1-4\sin^2\theta_W)^2(q\cdot p'q'\cdot p + q\cdot q'p'\cdot p) + q\cdot p'q'\cdot p - q\cdot q'p'\cdot p\Big] \qquad (3)
\end{aligned}
$$

Define

$$\xi = g_2^2(p+q)^2/(8e^2\cos^2\theta_W M_Z^2) = (p+q)^2/(8\sin^2\theta_W\cos^2\theta_W M_Z^2)$$

Then

$$
\begin{aligned}
&\langle|\mathcal{M}_\gamma+\mathcal{M}_Z|^2\rangle \approx \frac{8e^4}{(p+q)^4} \\
&\Big[q\cdot p'q'\cdot p(1+\xi(1+(1-4\sin^2\theta_W)^2 + q\cdot q'p'\cdot p(1+\xi((1-4\sin^2\theta_W)^2-1)\Big] \qquad (4)
\end{aligned}
$$

evaluating the invariants in the center of mass leads to

$$\langle|\mathcal{M}_\gamma+\mathcal{M}_Z|^2\rangle \approx e^4\left[(1+\cos^2\theta)(1+\xi(1-4\sin^2\theta_W)+2\xi\cos\theta\right] \qquad (5)$$

Finally, plugging into the cross section formula gives

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &\approx \frac{e^4}{64\pi^2 s}\left[(1+\cos^2\theta)\left(1+\xi(1-4\sin^2\theta_W)^2\right)+2\xi\cos\theta\right] \qquad (6) \\
\xi &\equiv -\frac{s}{8M_Z^2\sin^2\theta_W\cos^2\theta_W} = -\frac{G_F s}{4\pi\alpha\sqrt{2}} \qquad (7)
\end{aligned}
$$

b) Integrate over angles to find the total cross section σ and also find the front back asymmetry defined as

$$A_{FB} = \frac{1}{\sigma}\left[\int_{\theta<\pi/2} d\Omega - \int_{\theta>\pi/2} d\Omega\right]\frac{d\sigma}{d\Omega}.$$

Solution: We need:

$$
\begin{aligned}
&\int d\Omega = 4\pi, \qquad \int d\Omega\cos\theta = 0, \qquad \int d\Omega\cos^2\theta = 2\pi\int_{-1}^{1} z^2 dz = \frac{4\pi}{3} \\
&\int_{\theta<\pi/2} d\Omega\cos\theta = \pi \qquad (8)
\end{aligned}
$$

Then we get

$$\sigma \approx \frac{e^4}{12\pi s}\left(1+\xi(1-4\sin^2\theta_W)^2\right) \tag{9}$$

$$A_{FB} \approx \frac{3\xi/4}{1+\xi(1-4\sin^2\theta_W)^2} \approx \frac{3\xi}{4} = -\frac{3G_F s}{16\pi\alpha\sqrt{2}} = -6.74\times 10^{-5}\left[\frac{s}{GeV^2}\right] \tag{10}$$

where we used $\xi \ll 1$ in the last approximation.

12. Calculate the differential and total cross sections, with unpolarized electrons and muon spin unobserved, for the "flavor changing" neutrino scattering processes

a) $\nu_\mu + e^- \to \nu_e + \mu^-$

b) $\bar{\nu}_e + e^- \to \bar{\nu}_\mu + \mu^-$

In each case confirm the applicable one of Eqs(28.60) through (28.63) in the lecture notes.

Solution: Let initial neutrino momentum be q and the final neutrino momentum q', and the initial electron momentum p and the final muon be p'. Then the Feynman amplitudes fr the two processes are

$$\begin{aligned}\mathcal{M}_a &= \frac{G_F}{\sqrt{2}}\bar{u}(p')\gamma_\lambda(1-\gamma_5)u(q)\bar{u}(q')\gamma_\lambda(1-\gamma_5)u(p)\\ \mathcal{M}_b &= \frac{G_F}{\sqrt{2}}\bar{u}(p')\gamma_\lambda(1-\gamma_5)v(q')\bar{u}(q)\gamma_\lambda(1-\gamma_5)u(p)\end{aligned} \tag{11}$$

The squared amplitudes summed over final soins and averaged over initial spins are

$$\begin{aligned}\langle|\mathcal{M}_a|^2\rangle &= \frac{G_F^2}{4}\mathrm{Tr}(-\gamma\cdot p')\gamma_\lambda(1-\gamma_5)(-q\cdot\gamma)\gamma_\kappa(1-\gamma_5)\mathrm{Tr}(-q'\cdot\gamma)\gamma^\lambda(1-\gamma_5)(-\gamma\cdot p)\gamma^\kappa(1-\gamma_5)\\ \langle|\mathcal{M}_b|^2\rangle &= \frac{G_F^2}{4}\mathrm{Tr}(-\gamma\cdot p')\gamma_\lambda(1-\gamma_5)(-q'\cdot\gamma)\gamma_\kappa(1-\gamma_5)\mathrm{Tr}(-q\cdot\gamma)\gamma^\lambda(1-\gamma_5)(-\gamma\cdot p)\gamma^\kappa(1-\gamma_5)\end{aligned} \tag{12}$$

Note that crossing $q \leftrightarrow -q'$ relates the two processes. Each trace evaluation is similar:

$$\begin{aligned}\mathrm{Tr}(-\gamma\cdot p')\gamma_\lambda(1-\gamma_5)(-q\cdot\gamma)\gamma_\kappa(1-\gamma_5) &= 2\mathrm{Tr}(\gamma\cdot p')\gamma_\lambda(q\cdot\gamma)\gamma_\kappa(1-\gamma_5)\\ &= 8(p'_\lambda q_\kappa + q_\lambda p'_\kappa - \eta_{\kappa\lambda}p'\cdot q + i\epsilon_{\mu\lambda\kappa\nu}p'^\mu q^\nu)\end{aligned} \tag{13}$$

and similarly with the other traces. Then we have

$$\begin{aligned}\langle|\mathcal{M}_a|^2\rangle &= 16G_F^2\left(p'_\lambda q_\kappa + q_\lambda p'_\kappa - \eta_{\kappa\lambda}p'\cdot q + i\epsilon_{\mu\lambda\nu\kappa}p'^\mu q^\nu)\right)\left(p^\lambda q'^\kappa + q'^\lambda p^\kappa - \eta^{\kappa\lambda}p\cdot q' + i\epsilon^{\rho\lambda\sigma\kappa}q'_\rho p_\sigma)\right)\\ &= 16G_F^2\left(2p\cdot p'q\cdot q' + 2p\cdot qp'\cdot q' + +2p\cdot qp'\cdot q' - 2p\cdot p'q\cdot q'\right)\\ &= 64G_F^2 p\cdot qp'\cdot q'\end{aligned} \tag{14}$$

by crossing it follows that

$$\langle|\mathcal{M}_b|^2\rangle = 64G_F^2 p\cdot q'p'\cdot q \tag{15}$$

The Mandelstam invariants are

$$\begin{aligned} s &= -(p+q)^2 = m_e^2 - 2p\cdot q = m_\mu^2 - 2p'\cdot q' \\ u &= -(p+q')^2 = m_e^2 - 2p\cdot q' = m_\mu^2 - 2p'\cdot q \end{aligned} \tag{16}$$

so we can weite

$$\begin{aligned} \langle|\mathcal{M}_a|^2\rangle &= 16G_F^2(s-m_e^2)(s-m_\mu^2) \\ \langle|\mathcal{M}_b|^2\rangle &= 16G_F^2(u-m_e^2)(u-m_\mu^2) \end{aligned}$$

The differential cross section in the center of mass system is

$$\frac{d\sigma}{d\Omega} = \frac{p_\mu}{p_e}\frac{|\mathcal{M}|^2}{64\pi^2} \tag{17}$$

Since the neutrinos are assumed to ve massless, the ratio $p_\mu/p_e = (s-m_\mu^2)/(s-m_e^2)$. Then the cross sections are

$$\begin{aligned} \frac{d\sigma}{d\Omega}_a &= \frac{G_F^2(s-m_\mu^2)^2}{4\pi^2 s} \sim \frac{G_F^2 s}{4\pi^2} \\ \frac{d\sigma}{d\Omega}_b &= \frac{G_F^2(s-m_\mu^2)(u-m_\mu^2)(u-m_e^2)}{4\pi^2 s(s-m_e^2)} \sim \frac{G_F^2 u^2}{4\pi^2 s} \end{aligned}$$

where the last forms are for s much larger than the squared masses. We can relate u to the scattering angle:

$$u = m_e^2 + 2\boldsymbol{q}'\cdot\boldsymbol{p} - 2q'E_e = m_e^2 - 2q'(p\cos\theta + E_e) \sim -\frac{s}{2}(1+\cos\theta) \tag{18}$$

where $\boldsymbol{q}'\cdot\boldsymbol{p} = -\boldsymbol{q}'\cdot\boldsymbol{q} = -q'q\cos\theta$, so θ is the angle between the final neutrino momentum and the initial neutrino momentum. At large s the total cross sections are

$$\sigma_a \sim \frac{G_F^2 s}{\pi}, \qquad \sigma_b \sim \frac{G_F^2 s}{3\pi} \tag{19}$$