Standard Model/Quantun Field Theory III

Solution Set 5

Due: Wednesday, 25 March 2030

Suggested reading: QFT Notes, Ch 29; Sr, Secs 88-90; P, Ch 21; Sc, Ch 31. Here Sr=Srednicki, P=Peskin&Schroeder, and Sc=Schwartz.

13. Let I_1, I_2, I_3 be the generators for isospin in the vector representation. Prove the identity

$$e^{i\theta I_2} = 1 + iI_2 sin\theta + I_2^2(\cos\theta - 1) \tag{1}$$

by checking it in a basis where I_2 is diagonal. Use it to show that G-parity $G = Ce^{i\pi I_2}$ reverses the sign of all three components of the pion field. .

Solution: The eigenvalues of any I_k in the vector representation are $m = 1.0. - 1$. And since the representation is three dimensional there is precisely one eigenvector $|1, m\rangle$ belonging to each of the three eigenvalues, and these three form a basis. To check the identity we show that it is true on each basis vector. $e^{i\theta I_2}|1, m\rangle = e^{im\theta}|1, m\rangle$. On the other hand

$$(1 + iI_2 sin\theta + I_2^2(\cos\theta - 1))|1, m\rangle = (1 + im\sin\theta + m^2(\cos\theta - 1))|1, m\rangle \tag{2}$$

The right side is $\cos\theta + i\sin\theta = e^{i\theta}$, 1, $\cos\theta - i\sin\theta = e^{-i\theta}$, for $m = 1, 0, -1$ respectively, which establishes the identity. $e^{i\pi I_2}$ rotates a vector 180° about the 2 axis, so that $\pi_1, \pi_2, \pi_3 \to -\pi_1, \pi_2, -\pi_3$. On the other hand, from our definition, C takes $\pi_0 = \pi_3 \to \pi_0$ and $\pi_\pm \to -\pi_\mp$ where $\pi_\pm = \mp\pi_1 - i\pi_2$ So C reverses π_2 and leaves $\pi_{1,3}$ invariant. Thus G reverses all three components of $\boldsymbol{\pi}$

14. There is more to isospin invariance than mass degeneracies. The Δ is a prominent resonance in both π^+p and π^-p scattering, that has been determined to have isospin $I = 3/2$. Since the nucleon has $I = 1/2$ and the pion has $I = 1$, the pion nucleon system can have $I = 1/2, 3/2$.

a) Following the familiar angular momentum addition rules express the states $|\pi^+p\rangle, |\pi^-p\rangle, |\pi^0n\rangle$ in terms of total isospin states $|I, I_3\rangle$, for $I = 3/2, 1/2$.

Solution: The first step is to identify $|3/2, 3/2\rangle = |\pi^+p\rangle$. Then compute

$$\begin{aligned}
|3/2, 1/2\rangle\sqrt{\frac{15}{4} - \frac{3}{4}} &= (I_-^\pi + I_-^p)|\pi^+p\rangle = |\pi^0p\rangle\sqrt{2} + |\pi^+n\rangle \\
|3/2, 1/2\rangle &= \frac{1}{\sqrt{3}}\left[|\pi^0p\rangle\sqrt{2} + |\pi^+n\rangle\right] \\
|3/2, -1/2\rangle &= \frac{1}{\sqrt{3}}\left[|\pi^0n\rangle\sqrt{2} + |\pi^-p\rangle\right] \\
|3/2, -3/2\rangle &= |\pi^-n\rangle
\end{aligned} \tag{3}$$

Then $|1/2, 1/2\rangle$ is determined by orthogonality with $|3/2, 1/2\rangle$,

$$\begin{aligned}
|1/2, 1/2\rangle &= \frac{1}{\sqrt{3}}\left[|\pi^+n\rangle\sqrt{2} - |\pi^0p\rangle\right] \\
|1/2, -1/2\rangle &= \frac{1}{\sqrt{3}}\left[-|\pi^-p\rangle\sqrt{2} + |\pi^0n\rangle\right]
\end{aligned} \tag{4}$$

Finally we solve these equations for

$$\begin{aligned}
|\pi^0 n\rangle &= \frac{1}{\sqrt{3}}\left[|3/2,-1/2\rangle\sqrt{2}+|1/2,-1/2\rangle\right] \\
|\pi^- p\rangle &= \frac{1}{\sqrt{3}}\left[|3/2,-1/2\rangle-|1/2,-1/2\rangle\sqrt{2}\right] \\
|\pi^+ n\rangle &= \frac{1}{\sqrt{3}}\left[|3/2,-1/2\rangle+|1/2,-1/2\rangle\sqrt{2}\right] \\
|\pi^0 p\rangle &= \frac{1}{\sqrt{3}}\left[|3/2,1/2\rangle\sqrt{2}-|1/2,1/2\rangle\right]
\end{aligned}$$

b) Let $A_{3/2}$ and $A_{1/2}$ be the pion-nucleon scattering amplitudes in isospin 3/2, 1/2 respectively. Express the amplitudes for the processes $\pi^+ p \to \pi^+ p$, $\pi^- p \to \pi^- p$, $\pi^- p \to \pi_0 n$ in terms of the A's.

Solution: Using the relations obtained in part a) we read off:

$$\begin{aligned}
\langle \pi^+ p|\mathcal{M}|\pi^+ p\rangle &= A_{3/2} \\
\langle \pi^- p|\mathcal{M}|\pi^- p\rangle &= \frac{1}{3}A_{3/2}+\frac{2}{3}A_{1/2} \\
\langle \pi^0 n|\mathcal{M}|\pi^- p\rangle &= \frac{\sqrt{2}}{3}A_{3/2}-\frac{\sqrt{2}}{3}A_{1/2}
\end{aligned} \tag{5}$$

c) Use the results of part b) to predict the ratio of total cross sections for $\pi^+ p$ and $\pi^- p$ scattering (in an energy region where we can neglect multiple pion production)

$$\frac{\sigma_{\pi^+ p}}{\sigma_{\pi^- p}} = \frac{3\int d\Omega |A_{3/2}|^2}{\int d\Omega |A_{3/2}|^2 + 2\int d\Omega |A_{1/2}|^2}. \tag{6}$$

The $I = 3/2$ assignment to the Δ means that in the resonance energy region we should have $|A_{1/2}| \ll |A_{3/2}|$, which predicts a ratio of 3. Compare this prediction to the data.

Solution: In the isospin symmetry limit all pions have the same mass and $m_p = m_n$, so in the region in which only πN are in the final state the total cross sections are a comon factor times the integral over angles of the squared amplitudes and aummed over final states:

$$\begin{aligned}
\sigma_{\pi^+ p} &= K(s)\int d\Omega |A_{3/2}|^2 \\
\sigma_{\pi^- p} &= K(s)\int d\Omega \left[\frac{1}{9}|A_{3/2}+2A_{1/2}|^2+\frac{2}{9}|A_{3/2}-A_{1/2}|^2\right] \\
&= K(s)\int d\Omega \left[\frac{1}{3}|A_{3/2}|^2+\frac{2}{3}|A_{1/2}|^2\right]
\end{aligned} \tag{7}$$

Thus

$$\frac{\sigma_{\pi^+ p}}{\sigma_{\pi^- p}} = \frac{3\int d\Omega |A_{3/2}|^2}{\int d\Omega |A_{3/2}|^2 + 2\int d\Omega |A_{1/2}|^2}. \tag{8}$$

The $\pi^+ p$ total cross section is about 195 mb at the Δ resonance peak and the $\pi^- p$ cross section is about 68 mb at the same energy. The ratio is about 2.86, which is within 5% of the predicted factor of 3 from the assignment of $I = 3/2$ to the resonance.

15. π^-, K^- **beta decay** The solution to this problem will be given in the solutions to Set 6.