Standard Model/Quantun Field Theory III

Solution Set 6

Due: Wednesday, 8 April 2020

Suggested reading: QFT Notes, Ch 28.6-28.8; Sr, Secs 93-94; P, Ch 21; Sc, Ch 29.5. Here Sr=Srednicki, P=Peskin&Schroeder, and Sc=Schwartz.

15. This exercise is to derive the phase space costraints (29.23) in the lecture notes, which follow from the Nambu proposal Action $= T_0\times$ (Area).

a) Parametrize the worldsheet with σ, τ, and define $\dot{x}^\mu = \partial x^\mu/\partial\tau$ and $x'^\mu = \partial x^\mu/\partial\sigma$. Argue that the "area" of the worldsheet can be taken as

$$A = \int d\tau d\sigma \left\{(\dot{x}\cdot x')^2 - \dot{x}^2 x'^2\right\}^{1/2} \tag{1}$$

You may find the formula for the directed area given in Eq(21.33) a useful starting point.

Solution;The area element in the $\mu\nu$ plane is

$$d\sigma^{\mu\nu} = d\sigma d\tau \left(\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau} - \frac{\partial x^\nu}{\partial\sigma}\frac{\partial x^\mu}{\partial\tau}\right) \tag{2}$$

The squared invariant area is then

$$dA^2 = \frac{1}{2}d\sigma^{\mu\nu}d\sigma_{\mu\nu} = (d\sigma d\tau)^2\left(\dot{x}^2 x'^2 - (\dot{x}\cdot x')^2\right) \tag{3}$$

We take the square root of minus the right side, because we want the the area to be real for a timelike worldsheet ($dA^2 < 0$). The desired result follows.

b) Nambu's action is then $S = -T_0 A \equiv \int d\tau d\sigma \mathcal{L}(\dot{x}, x')$. Work out $\mathcal{P}_\mu = \partial\mathcal{L}/\partial\dot{x}^\mu$ and then derive the desired phase space constraints.

Solution; We first work out $\mathcal{P}$:

$$\mathcal{P}^\mu = T_0\frac{\dot{x}^\mu x'^2 - x^{\mu\prime}\dot{x}\cdot x'}{\left\{(\dot{x}\cdot x')^2 - \dot{x}^2 x'^2\right\}^{1/2}} \tag{4}$$

One immediately sees that $x'\cdot\mathcal{P} = 0$ because the two terms in the numerator then cabcel. Squaring $\mathcal{P}$ and using $x'\cdot\mathcal{P} = 0$ gives

$$\mathcal{P}^2 = T_0^2 x'^2\frac{\dot{x}^2 x'^2 - (\dot{x}\cdot x')^2}{\left\{(\dot{x}\cdot x')^2 - \dot{x}^2 x'^2\right\}} = -T_0^2 x'^2 \tag{5}$$

16. π^-, K^- **beta decay**

a) Exploiting the CVC hypothesis for pion beta decay $\pi^- \to \pi^0 + e^- + \bar{\nu}_e$ complete the calculation of the total rate. By comparing to experiment get an estimate of $|V_{ud}|$. For simplicity you may set $m_e = 0$.

Solution; According to the CVC hypothesis (exact isospin symmetry)

$$\langle \pi^0, p' | \bar{u}\gamma^\mu d | \pi^-, p\rangle \approx \frac{(p+p')^\mu}{(2\pi)^3\sqrt{4EE'}} F(q^2), \qquad q = p - p' \tag{6}$$

where $F(0) = \sqrt{2}$. Then the Feynman amplitude for beta decay is, using $F(q^2) \approx F(0) = \sqrt{2}$,

$$\begin{aligned} \mathcal{M} &\approx iG_F V_{ud} \bar{u}_e (p+p')\cdot\gamma(1-\gamma_5)v_{\bar{\nu}} \\ \sum |\mathcal{M}|^2 &= G_F^2 |V_{ud}|^2 \mathrm{Tr}(p+p')\cdot\gamma(1-\gamma_5)(-\gamma\cdot p_\nu)(1+\gamma_5)(p+p')\cdot\gamma(m_e - \gamma\cdot p_e) \\ &= G_F^2 |V_{ud}|^2 \mathrm{Tr}(p+p')\cdot\gamma(1-\gamma_5)\gamma\cdot p_\nu(1+\gamma_5)(p+p')\cdot\gamma\gamma\cdot p_e \end{aligned} \tag{7}$$

The integral over the $e\bar{\nu}$ phase space, with $m_e = 0$, can be taken from the Lecture notes Eq. (26.30):

$$\Gamma = \frac{1}{2m_{\pi^-}} \int \frac{d^3p'}{(2\pi)^3 2E'} (p+p')_\mu (p+p')_\nu \frac{q^\mu q^\nu - q^2 \eta^{\mu\nu}}{3\pi} \tag{8}$$

where $q = p - p'$. Calculating

$$\begin{aligned} q^2 &= -m_{\pi^0}^2 - m_{\pi^-}^2 + 2m_{\pi^-}E', \quad (p+p')^2 = -m_{\pi^0}^2 - m_{\pi^-}^2 - 2m_{\pi^-}E' \\ q\cdot(p+p') &= p^2 - p'^2 = m_{\pi^0}^2 - m_{\pi^-}^2 \end{aligned}$$

from which

$$\begin{aligned} \Gamma &= \frac{G_F^2 |V_{ud}|^2}{2m_{\pi^-}} \int \frac{d^3p'}{(2\pi)^3 2E'} \frac{4m_{\pi^-}^2 p'^2}{3\pi} = \frac{G_F^2 |V_{ud}|^2 m_{\pi^-}}{6\pi^3} \int_0^\Delta \frac{p'^4 dp'}{\sqrt{m_{\pi^0}^2 + p'^2}} \\ &= \frac{G_F^2 m_{\pi^0}^4 |V_{ud}|^2 m_{\pi^-}}{6\pi^3} \int_0^{\Delta/m_{\pi^0}} \frac{u^4 du}{\sqrt{1+u^2}} \end{aligned} \tag{9}$$

The upper limit Δ is determined by the requirement that q be timelike or $E' < (m_{\pi^-}^2 + m_{\pi^0}^2)/(2m_{\pi^-})$:

$$\Delta^2 = \frac{(m_{\pi^-}^2 - m_{\pi^0}^2)^2}{4m_{\pi^-}^2} \tag{10}$$

In the case of pion beta decay $\Delta \ll m_{\pi^0}$ so p' in the denominator can be dropped and the integral is trivial

$$\begin{aligned} \Gamma_{\pi^-\to\pi^0 e\bar{\nu}_e} &\approx \frac{G_F^2 |V_{ud}|^2}{30\pi^3} \frac{m_{\pi^-}}{m_{\pi^0}} (m_{\pi^-} - m_{\pi^0})^5 \left[\frac{m_{\pi^-} + m_{\pi^0}}{2m_{\pi^-}}\right]^5 \approx |V_{ud}|^2 \cdot 2.865 \times 10^{-22}\mathrm{MeV} \\ &\approx \frac{G_F^2 |V_{ud}|^2}{30\pi^3} (m_{\pi^-} - m_{\pi^0})^5 \approx |V_{ud}|^2 \cdot 3.011 \times 10^{-22}\mathrm{MeV} \\ &\approx |V_{ud}|^2 \cdot (0.435, 0.457)\hbar \mathrm{s}^{-1} \end{aligned} \tag{11}$$

The approximations here neglect $m_{\pi^-} - m_{\pi^0}$ in comparison to m_{π^-} an excellent approximation. The experimental rate is 0.398/s giving the estimate $|V_{ud}|^2 \approx 0.915 = (0.956)^2$. This is somewhat less than the accepted value $|V_{ud}| = 0.975$ mostly due to the approxmation to the intgeral and the neglect of the electron mass.

b) Kaon beta decay $K^- \to \pi^0 + e^- + \bar{\nu}_e$ involves similar kinematics, but of course involves V_{us} and the estimate of the matrix element based on CVC requires $SU(3)$ symmetry arguments that are less reliable than the isospin symmetry used for pion decay. Nonetheless calculate the rate based on the assumption of exact $SU(3)$ for the matrix element, and get an estimate of $|V_{us}|$ from the data.

Solution: To adapt the π^- calculation to K^-, we of course have to substitute $V_{ud} \to V_{us}$ and $m_{\pi^-} \to m_{K^-}$. Also the $SU(3)$ matrix elements must be used. For π^- we needed $f_{123} = \epsilon_{123} = 1$. But for K^- we need $f_{345} = 1/2$, so the rate has an additional factor of 1/4. Finally we cannot neglect $m_{K^-} - m_{\pi^0}$ in comparison to m_{K^-} so the phase space integral is more complicated:

$$\begin{aligned}\Gamma_{K^-\to\pi^0 e\bar{\nu}_e} &\approx \frac{1}{4}|V_{us}|^2 m_{K^-}\frac{G_F^2 m_{\pi^0}^4}{16\pi^3}\left[\ln\left(\frac{\Delta}{m_{\pi^0}} + \sqrt{1+\frac{\Delta^2}{m_{\pi^0}^2}}\right)\right. \\ &\qquad \left. -\frac{\Delta}{m_{\pi^0}}\sqrt{1+\frac{\Delta^2}{m_{\pi^0}^2}} + \frac{2\Delta^3}{3m_{\pi^0}^3}\sqrt{1+\frac{\Delta^2}{m_{\pi^0}^2}}\right] \\ &\approx |V_{us}|^2 m_{K^-}\frac{G_F^2 m_{\pi^0}^4}{64\pi^3}\left[\ln\frac{m_{K^-}}{m_{\pi^0}} - \frac{\Delta}{m_{\pi^0}}\sqrt{1+\frac{\Delta^2}{m_{\pi^0}^2}} + \frac{2\Delta^3}{3m_{\pi^0}^3}\sqrt{1+\frac{\Delta^2}{m_{\pi^0}^2}}\right] \quad (12)\end{aligned}$$

where $\Delta = (m_{K^-}^2 - m_{\pi^0}^2)/2m_{K^-} \approx 0.4626 m_{K^-} \approx 1.692 m_{\pi^0}$. for the process studied here. For this value the quantity in square brackets is ≈ 4.318 [for the π^- it was 2.267×10^{-8}]. The rate for kaon beta decay becomes

$$\Gamma_{K^-\to\pi^0 e\bar{\nu}_e} \approx |V_{us}|^2 4.849 \times 10^{-14}\text{MeV} \approx |V_{us}|^2 \times 7.368 * 10^7 \hbar\text{s}^{-1} \quad (13)$$

The experimental rate is $4.08 \times 10^6 \text{s}^{=1}$ leading to $|V_{us}| \approx 0.235$ compared to the consensus value of 0.22.

17. τ **Decay to** $\pi\nu_\tau$ **or** $K\nu_\tau$. The decays of the τ lepton into a pseudoscalar meson and a ν_τ are completely given, to lowest order in G_F, in terms of the meson decay constants as measured in the meson decay into electron or muon plus neutrino. Calculate the rates for $\tau \to \pi^- + \nu_\tau$ and $\tau \to K^- + \nu_\tau$. Compare your results with experiment.

Solution: Let q, p, p' be the momenta of $\pi (or K), \tau, \nu_\tau$ respectively. Then the Feynman amplitude for either of the processes is

$$\mathcal{M} = iF_k G_F V_{uk}^{CKM} q_\mu \bar{u}_{\nu_\tau}\gamma^\mu(1-\gamma_5)u_\tau = -im_\tau iF_K G_F V_{uk}^{CKM}\bar{u}_{\nu_\tau}(1+\gamma_5)u_\tau \quad (14)$$

where k standa for π or K, and the second form used $q = p - p'$ and the Dirac equation for the spinors. The squared amplitude is

$$\begin{aligned}|\mathcal{M}|^2 &= |F_k G_F V_{uk}^{CKM}|^2 \bar{u}_\tau(1-\gamma_5)(-p'\cdot\gamma(1+\gamma_5 u_\tau) \\ &= 4m_\tau^2|F_k G_F V_{uk}^{CKM}|^2(-p\cdot p') = 2m_\tau^2|F_k G_F V^{CKM} uk|^2(q^2+m_\tau^2) \\ &= 2m_\tau^2|F_l G_F V_{uk}^{CKM}|^2(m_\tau^2 - m_k^2) \quad (15)\end{aligned}$$

In the τ rest frame $m_\tau = p' + \sqrt{m_k^2 + p'^2}$ which gives $p' = (m_\tau^2 - m_k^2)/(2m_\tau)$. Plugging into the formula for the rate gives

$$\Gamma_{\tau\to k\nu_\tau} = \frac{G_F^2 f_k^2 |V_{uk}|^2}{8\pi m_\tau}(m_\tau^2 - m_k^2)^2 \tag{16}$$

For the pion, we use $m_\tau = 1.78\text{GeV}$, $m_\mu = 0.106\text{GeV}$. $m_\pi = 0.140\text{GeV}$, $F_\pi = 0.093\text{GeV}$, $V_{ud} = 0.975$, to get

$$\begin{aligned}
\Gamma_{\tau\to\pi\nu_\tau} &= \frac{G_F^2 F_\pi^2 |V_{ud}|^2}{8\pi m_\tau}(m_\tau^2 - m_\pi^2)^2 \approx 2.4358\times 10^{-10}\text{MeV} = 0.3701\times 10^{12}\hbar\text{s}^{-1} \\
\text{B.F.} &= \tau_\tau \Gamma_{\tau\to\pi\nu_\tau} \approx 0.1077 \qquad \text{Exp} : 0.108 \pm .005
\end{aligned}$$

For the kaon we use $V_{us} = 0.22$. $F_K = 0.113\text{GeV}$ to calculate the ratio

$$\begin{aligned}
\frac{\Gamma_{\tau\to K\nu_\tau}}{\Gamma_{\tau\to\pi\nu_\tau}} &= \frac{f_K^2 |V_{us}|^2}{f_\pi^2 |V_{ud}|^2}\frac{(m_\tau^2 - m_K^2)^2}{(m_\tau^2 - m_\pi^2)^2} \approx .0655 \quad \text{Exp} : 0.0644 \\
&= \frac{\Gamma_{K\to\mu\nu_\mu}}{\Gamma_{\pi\to\mu\nu_\mu}}\frac{m_K^3 (m_\pi^2 - m_\mu^2)^2}{m_\pi^3 (m_K^2 - m_\mu^2)^2}\frac{(m_\tau^2 - m_K^2)^2}{(m_\tau^2 - m_\pi^2)^2} \approx 0.0625
\end{aligned} \tag{17}$$

The second line uses the experimental value for Γ_K/Γ_π instead of separate values for F and V.