Standard Model/Quantun Field Theory III

Solution Set 7

Due: Wednesday, 22 April 2020

Suggested reading: QFT Notes, Ch 29.5-29.6 and 30; Sr, Secs 93-94; P, Ch 21; Sc, Ch 29.5. Here Sr=Srednicki, P=Peskin&Schroeder, and Sc=Schwartz.

18. **Nonleptonic baryon decay** Test the $\Delta I = 1/2$ rule for the isoscalar Λ baryon decays into $p\pi^-$ and $n\pi_0$ by expressing their rates in terms of $I = 1/2$ and $I = 3/2$ amplitudes. Compare the ratio of rates calculated with the assumption of pure $\Delta I = 1/2$ transitions to the ratio extracted from the data.

Solution: We can take the isospin decomposirions from Problem 9 of set 4.

$$\begin{aligned} |\pi^0 n\rangle &= \frac{1}{\sqrt{3}}\left[|3/2,-1/2\rangle\sqrt{2} + |1/2,-1/2\rangle\right] \\ |\pi^- p\rangle &= \frac{1}{\sqrt{3}}\left[|3/2,-1/2\rangle - |1/2,-1/2\rangle\sqrt{2}\right] \end{aligned}$$

Then

$$\frac{|\langle\pi^0 n|\Lambda\rangle|^2}{|\langle\pi^- p|\Lambda\rangle|^2} = \frac{|A_{3/2}\sqrt{2} + A_{1/2}|^2}{|A_{3/2} - A_{1/2}\sqrt{2}|^2} \to \frac{1}{2}, \qquad |A_{3/2}| \ll |A_{1/2}| \tag{1}$$

19. Fill in some gaps in our discussion of the box diagram model of neutral kaon mixing (Section 29.5.2 of the lecture notes).

a) Calculate the integral encountered in our discussion of $K, \bar{K}$

$$A(x_i, x_j) = \frac{M_W^2}{\pi^2}\int \frac{d^4 k_E}{(k_E^2 + M_W^2)^2}\frac{k_E^2/D}{(k_E^2 + m_i^2)(k_E^2 + m_j^2)} \tag{2}$$

to establish the last line of Eq.(29.83) in the lecture notes. Here $x_i = m_i^2/M_W^2$.

Solution: This integral is convergent in $D = 4$; the integral over angles at $D = 4$ is just $2\pi^2$. We can also scale out M_W, $k_E = M_W u$ so the integral reduces to

$$A(x_i, x_j) = \frac{1}{2}\int_0^\infty \frac{u^5 du}{(u^2+1)^2}\frac{1}{(u^2+x_i)(u^2+x_j)} = \frac{1}{4}\int_0^\infty \frac{v^2 dv}{(v+1)^2}\frac{1}{(v+x_i)(v+x_j)} \tag{3}$$

The integral is easily done by expanding the integrand in partial fractionis

$$\mathcal{I} = \frac{1}{(v+1)^2}\frac{1}{(1-x_i)(1-x_j)} + \frac{B}{v+1} + \frac{1}{v+x_i}\frac{x_i^2}{(x_j-x_i)(1-x_i)^2} + \frac{1}{v+x_j}\frac{x_j^2}{(x_i-x_j)(1-x_j)^2}$$

where B is determined by the requirement that $\mathcal{I} \sim v^{-2}$ at infinity. Then integating term by term we get

$$A_{ij} = \frac{1}{4}\left[\frac{1}{(1-x_i)(1-x_j)} - \frac{x_i^2 \ln x_i}{(x_j-x_i)(1-x_i)^2} - \frac{x_j^2 \ln x_j}{(x_i-x_j)(1-x_j)^2}\right] \tag{4}$$

as desired.

b) Derive the approximate formula (Eq.(29.86))

$$\sum_{i,j} \xi_i \xi_j A_{i,j} = \xi_u^2 x_c + \xi_t^2 \left(\frac{x_t + x_t^2}{(1-x_t)^2} + \frac{2x_t \ln x_t}{(1-x_t)^3} \right) + 2\xi_u \xi_t x_c \left(\frac{x_t}{(1-x_t)} + \frac{\ln x_t}{(1-x_t)^2} - \ln x_t \right) \quad (5)$$

Solution: We first obtain Eq. (28.79):

$$\begin{aligned} \sum_{i,j} \xi_i \xi_j A_{ij} &= \xi_u^2 A_{uu} + \xi_t^2 A_{tt} + (\xi_u + \xi_t)^2 A_{cc} + 2\xi_u \xi_t A_{ut} - 2\xi_u(\xi_u + \xi_t) A_{uc} - 2\xi_t(\xi_u + \xi_t) A_{ct} \\ &= \xi_u^2 (A_{uu} + A_{cc} - 2A_{uc}) + \xi_t^2 (A_{tt} + A_{cc} - 2A_{ct}) \\ &\quad + 2\xi_u \xi_t (A_{cc} + A_{ut} - A_{ct} - A_{uc}) \end{aligned} \quad (6)$$

Next we find A_{ii} by taking the limit $x_j \to x_i$

$$\begin{aligned} A_{ii} &= \left(\frac{x_i^2 \ln x_i}{(x_i - 1)^2} \right)' + \frac{1}{(x_i-1)^2} \\ &= \frac{2x_i \ln x_i + x_i}{(x_i-1)^2} - 2\frac{x_i^2 \ln x_i}{(x_i-1)^3} + \frac{1}{(x_i-1)^2} \\ &= \frac{-2x_i \ln x_i + x_i^2 - 1}{(x_i-1)^3} \end{aligned} \quad (7)$$

Setting $x_u = 0$ and dropping terms of order $x_c^2, x_c^2 \ln x_c$ gives

$$\begin{aligned} & A_{uu} = 1, \qquad A_{uc} \approx 1 + x_c(1 + \ln x_c), \qquad A_{ut} = \frac{1 - x_t(1 - \ln x_t)}{(1-x_t)^2} \\ & A_{cc} \approx 1 + 2x_c \ln x_c + 3x_c, \qquad A_{ct} \approx \frac{1 - x_t + x_t \ln x_t}{(x_t-1)^2} + x_c \frac{1 - x_t + \ln x_t}{(x_t-1)^2} \\ & A_{tt} = \frac{-2x_t \ln x_t + x_t^2 - 1}{(x_t-1)^3} \end{aligned} \quad (8)$$

Then assembling the coefficients

$$\begin{aligned} A_{uu} + A_{cc} - 2A_{uc} &\approx x_c \\ A_{cc} + A_{ut} - A_{ct} - A_{uc} &\approx x_c \left[\ln x_c + 2 - \frac{1 - x_t + \ln x_t}{(x_t-1)^2} \right] \\ A_{tt} + A_{cc} - 2A_{ct} &\approx \frac{-2x_t \ln x_t + x_t^2 - 1}{(x_t-1)^3} + 1 - 2\frac{1 - x_t + x_t \ln x_t}{(x_t-1)^2} \\ &\quad + 2x_c \ln x_c + 3x_c - 2x_c \frac{1 - x_t + \ln x_t}{(x_t-1)^2} \\ &\approx \frac{-2x_t^2 \ln x_t}{(x_t-1)^3} + \frac{x_t^2 + x_t}{(1-x_t)^2} + x_c \left[2\ln x_c + 3 - 2\frac{1 - x_t + \ln x_t}{(x_t-1)^2} \right] \end{aligned} \quad (9)$$

Since ξ_t is exceedingly small, it makes sense to drop terms of order $x_c \xi_t^2$, in which case these ciefficients reproduce Eq.(28.80) after a typo in the latter is corrected: the $\ln x_t$ inthe coefficient of ξ_t^2 should be multiplied by x_t^2 not by x_t.

c) Put in the experimental numbers and check Eq.(28.88).

Solution: Putting in the experimental numbers for $\xi_u \approx .22$, $G_f \approx 1.1 \times 10^{-5}\text{GeV}^{-2}$, $m_c \approx 1.3\text{GeV}$, $m_{K^0} \approx 498\text{MeV}$, $F_K \approx 0.1\text{GeV}$, we get

$$h' \approx (0.22 \times 1.1 \times 1.35 \times 0.1)^2 \times 498/(6\pi^2) \times 10^{-10}\text{MeV} \approx 0.9 \times 10^{-12}\text{MeV} \tag{10}$$

20. Deep inelastic scattering is the process $e+p \to e+X$ where X represents an arbitrary hadronic state. Only the final electron is observed, meaning that the squared amplotude is summed over all possible X. We work to lowest order in QED perturbation theory but nonperturbatively in the strong interactions (QCD). The Feynman amplitude for a fixed X can be expressed as the product of the electron spinor bilinear $\bar{u}'\gamma^\mu u$ and the matrix element of the electromagnetic current between X and the proton, $\langle X|j_\mu(0)|p\rangle$. Of course the familiar factors $1/[(2\pi)^{3/2}\sqrt{2E_i}$ for each external particlw in the matrix element are dropped in forming the Feynman amplitude. The squared Feynman amplitude averaged over proton spins and summed over all X can be written as

$$\frac{e^4}{q^4}\bar{u}'\gamma^\mu u\bar{u}\gamma^\nu u' W_{\mu\nu}(q,p) \tag{11}$$

where p is the proton momentum, $q = k - k'$ is the momentum transferred from the electron to the proton.

a) Show that Lorentz covariance and current conservation determine W in terms of two structure functions of q^2 and $q \cdot p$:

$$\begin{aligned} W^{\mu\nu} &= \left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) W_1(q^2,\nu) \\ &\quad + \left(p^\mu - q^\mu\frac{q\cdot p}{q^2}\right)\left(p^\nu - q^\nu\frac{q\cdot p}{q^2}\right) W_2(q^2,\nu) \end{aligned} \tag{12}$$

where $m_p\nu = -q \cdot p$.

Solution: First of all the right side of the above equation is indeed Lorentz covariant. It also satisfies current conservation since the tensor structures of each term give 0 when dotted into q on either index. So what needs to be shown is that there are no further terms with thexe properties. So consider the tensor structures

$$Aq^\mu q^\nu + Bp^\mu q^\nu + Cp^\nu q^\mu \tag{13}$$

which are not present on the right side. This structure is Lorentz covariant if A, B, C are any functions of the invariants. So we ask is it conserved? First dot the μ index with q and set it to zero

$$Aq^2 q^\nu + Bq \cdot pq^\nu + Cq^2 p^\nu = 0 \tag{14}$$

Since p and q point in different directions, this implies $C = 0$ and $Aq^2 + Bq \cdot p = 0$. Doing the same with the ν index implies $B = 0$ and $Aq^2 + Cq \cdot p = 0$. All together then conservation implies $A = B = C = 0$. That is the above form is indeed the most general one.

b) Calculate the differential cross section, summed and averaged over electron spins, for deep inelastic electron scattering in the proton rest frame. Express your answer in terms of the structure functions $W_{1,2}$, the initial electron energy E_e, and the electron scattering angle θ.

Solution: Since the electron spinor bilinear is conserved $\bar{u}' q \cdot \gamma u = 0$ we can drop all of the terms in $W_{\mu\nu}$ with either a q^μ or q^ν or both. Thus the squared Feynman amplitude, summed over all final states X is

$$\sum |\mathcal{M}|^2 = \frac{e^4}{q^4} \left[\bar{u}' \gamma^\mu u \bar{u} \gamma_\mu u' W_1 + [\bar{u}' p \cdot \gamma u \bar{u} p \cdot \gamma u' W_2 \right] \tag{15}$$

Summing over final electron spins and averaging initial spins yields

$$\begin{aligned} \frac{1}{2} \sum \bar{u}' \gamma^\mu u \bar{u} \gamma_\mu u' &= \frac{1}{2} \mathrm{Tr}(m - \gamma \cdot k') \gamma^\mu (1 - k \cdot \gamma) \\ &= -2m^2 \eta^{\mu\nu} + 2(k'^\mu k^\nu + k'^\nu k^\mu - \eta^{\mu\nu} k \cdot k') \end{aligned} \tag{16}$$

and then

$$\begin{aligned} \sum |\mathcal{M}|^2 &= \frac{e^4}{q^4} \left[(-8m^2 - 4k \cdot k') W_1 + 2m_p^2 (m^2 + k \cdot k')) + 4m_p^2 E_e E_e') W_2 \right] \\ &= \frac{e^4}{q^4} \left[(-8m^2 - 4(kk' \cos\theta - E_e E_e')) W_1 + 2m_p^2 (m^2 + kk' \cos\theta + E_e E_e') W_2 \right] \end{aligned} \tag{17}$$

Neglecting the electron mass, and so setting $E_e, E_e' = k, k'$ allows the simplification

$$\sum |\mathcal{M}|^2 \approx \frac{e^4}{q^4} [4kk' \cos^2 \frac{\theta}{2} [m_p^2 W_2 + 2W_1 \tan^2 \frac{\theta}{2}]$$

Inserting this result into the formula for the cross section, we find

$$\begin{aligned} d\sigma &= \frac{d^3k'}{(2\pi)^3 2E_e'} \frac{|\mathcal{M}|^2}{4E_e m_p v} \\ &\approx \frac{k'^2 dk' d\Omega}{64\pi^3 E_e' m_p k} \frac{e^4}{16(kk')^2 \sin^4(\theta/2)} [4kk' \cos^2 \frac{\theta}{2}][m_p^2 W_2 + 2W_1 \tan^2 \frac{\theta}{2}] \\ &\approx e^4 \frac{d\nu d\Omega}{256\pi^3 m_p k^2} \frac{\cos^2(\theta/2)}{\sin^4(\theta/2)} [m_p^2 W_2 + 2W_1 \tan^2 \frac{\theta}{2}] \end{aligned}$$