Notes on Curvature in two dimensions

The parallel postulate of Euclidean geometry states that given a straight line \( L \) and a point \( p \) not on \( L \), there is precisely one straight line which intersects \( p \) and does not intersect \( L \). It follows, that the perpendicular distance between two straight lines is constant if they do not intersect.

On a manifold with curvature, we expect this postulate to hold no longer. In particular given two “straight” lines which are initially parallel, the distance between them might change. In this context a line is “straight” if it is a geodesic and is, therefore, a curve which extremizes the distance between two points on the line.

As an example, consider line \( L \) to be the \( x \)-axis. Let a point \( p \) be a small perpendicular distance \( \xi \) away from \( L \), and consider a straight line \( L' \) which intersects \( p \). Assume that the perpendicular distance \( \xi(x) \) is given by

\[
\xi(x) = a + bx + \frac{1}{2}cx^2 + \frac{1}{6}dx^3 + \ldots
\]

for constants \( a, b, \ldots \) If the manifold is flat and we are using Cartesian coordinates, then we expect that \( c, d \) and the constant coefficients in all higher order terms are zero. In particular, we note that

\[
d\xi(x)/dx = b
\]

and

\[
d^2\xi(x)/dx^2 = 0.
\]

This is the hallmark of a flat manifold: The perpendicular distance between two geodesics changes linearly in the distance along one geodesic.

Similarly, we expect that if a manifold is not flat, then the perpendicular distance between two geodesics changes in a more complicated manner.

For a two dimensional manifold, we can define the scalar curvature \( R \) at a point \( p \) on the manifold from the equation

\[
\frac{d^2\xi(x)}{dx^2} = -R\xi
\]

where \( p \) is a point on a geodesic \( L \), parameterized by distance \( x \), and \( \xi(x) \) is the infinitesimal distance to any nearby geodesic \( L' \). For a given \( p \), it should be modestly surprising that \( R \) is unique, and independent of which geodesics are being considered. Later, we will see why this is so. We will also derive a slightly different version of the previous equation, called the equation of geodesic deviation.

As an example consider the geometry of a two-sphere of radius \( a \), and let geodesic \( L \) be a curve of constant \( \phi = 0 \) while \( \theta = 0, \pi \), and let \( L' \) be a curve of constant \( \phi = \delta\phi \) while \( \theta = 0, \pi \). \( \delta\phi \) is assumed to be infinitesimal. The distance from the axis, along one of these geodesics is \( x = a\theta \). The distance between these two geodesics is \( \xi(x) = a\delta\phi \sin\theta = a\delta\phi \sin(x/a) \). We calculate

\[
\frac{d^2\xi(x)}{dx^2} = -\frac{\delta\phi}{a} \sin(x/a) = -\frac{1}{a^2}\xi(x).
\]
We conclude that the scalar curvature of a two-sphere is \( R = 1/a^2 \).

For a second example, let the two dimensional manifold consist of time \( t \), and radius \( r \) from the center of the Earth of mass \( M \). The claim is made that objects in free-fall move along geodesics of curved spacetime. If this is so, then we should be able to measure the scalar curvature of spacetime by observing objects in free-fall. Consider two objects, \( m_1 \) and \( m_2 \), which are initially at rest at \( r_1 \) and \( r_1 + \delta r \). Let the two objects be released simultaneously. \( m_1 \) falls at a rate governed by

\[
\frac{d^2 r_1}{dt^2} = -\frac{GM}{r_1^2}
\]

whereas \( m_2 \) falls at a rate given by

\[
\frac{d^2 (r_1 + \delta r)}{dt^2} = -\frac{GM}{(r_1 + \delta r)^2} \approx -\frac{GM(1 - 2\delta r/r_1)}{r_1^2},
\]

through first order in \( \delta r \). Subtracting these equations results in

\[
\frac{d^2 \delta r}{dt^2} = \frac{2GM\delta r}{r_1^3}.
\]

We conclude that the scalar curvature of this manifold is \( R = 2GM/r^3 \), after I change the sign to account for the fact that having time as one of our coordinates requires a slightly different definition of scalar curvature.

WARNING: This description of curvature is not intended to be complete and, specifically, is appropriate only for a two dimensional manifold.