Jackson says “By manipulation of the power series solutions it is possible to obtain a compact representation of the Legendre polynomials known as Rodrigues’ formula.” Here is a proof that Rodrigues’ formula indeed produces a solution to Legendre’s differential equation. From the differential equation, assuming a series solution
\[ P_n = \sum a_j x^j \quad (\alpha = 0) \]
we obtained the relation
\[ a_{j+2} = \frac{j(j+1) - n(n+1)}{(j+1)(j+2)} a_j \]
[JDJ (3.14), with \( \alpha = 0 \)]. With \( j = n - 2k \), this is satisfied by
\[ a_{n-2k} = (-1)^k \frac{(2n - 2k)!}{2^n k! (n-k)! (n-2k)!}, \]
where \( 1/2^n \) is conventional. So, we can write
\[ P_n(x) = \sum_{k=0}^\lfloor n/2 \rfloor (-1)^k \frac{(2n - 2k)!}{2^n k! (n-k)! (n-2k)!} x^{n-2k}, \]
where \( \lfloor n/2 \rfloor \) denotes the “greatest integer” or the integer part. For integer \( n - 2k \), this is
\[ P_n(x) = \sum_{k=0}^{n/2} (-1)^k \frac{(2n - 2k)!}{2^n k! (n-k)! (n-2k)!} x^{n-2k}, \]
where the extra terms introduced by extending the upper limit of the sum from \( \lfloor n/2 \rfloor \) to \( n \) have zero derivative. By the binomial theorem, this expression is
\[ P_n(x) = \frac{1}{2^n n!} \left( \frac{d}{dx} \right)^n (x^2 - 1)^n. \quad (3.16) \]
Jackson next says, “From Rodrigues’ formula it is a straightforward matter” to obtain recursion relations for the \( P_n \). To this end, first prove some relations that are useful in many applications. Let \( D = d/dx \). Then, for any function \( f(x) \),
\[ D^l(xf) = x (D^l f) + l (D^{l-1} f). \]
This can be proved by induction: it clearly holds for \( l = 0 \), for which it reads \( xf = xf \), and for \( l = 1 \) by the product rule for derivatives, \( D(xf) = x(Df) + f(Dx) \). Suppose it holds for \( l - 1 \); then
\[ D^l(xf) = D[D^{l-1}(xf)] = D[x(D^{l-1} f + (l-1)D^{l-2} f)] \]
\[ = [x(D^l f) + D^{l-1} f] + (l-1)D^{l-1} f = x(D^l f) + l(D^{l-1} f). \]
Apply for $g(x) = xf(x)$:
\[
\mathcal{D}^l(x^2 f) = \mathcal{D}^l(x \cdot f) = x\mathcal{D}^l(f) + l\mathcal{D}^{l-1}(f)
\]
\[
= x[x(\mathcal{D}^l f) + l\mathcal{D}^{l-1}f] + l[x\mathcal{D}^{l-1}f + (l-1)\mathcal{D}^{l-2}f]
\]
\[
= x^2(\mathcal{D}^l f) + 2lx(\mathcal{D}^{l-1}f) + l(l-1)(\mathcal{D}^{l-2}f).
\]
This procedure iterated leads to the general and perhaps well known result
\[
\mathcal{D}^n(fg) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (\mathcal{D}^k f)(\mathcal{D}^{n-k}g),
\]
and so in particular
\[
\mathcal{D}^l[(x^2 - 1)f] = (x^2 - 1)(\mathcal{D}^l f) + 2lx(\mathcal{D}^{l-1}f) + l(l-1)(\mathcal{D}^{l-2}f).
\]*

Now, use (*) to prove the middle of Jackson’s (3.29). Apply $\mathcal{D}$ to Rodrigues’ formula for $P_{l+1}$, first commuting $\mathcal{D}^l$ as above and then applying the product rule for derivatives a number of times:
\[
\mathcal{D}P_{l+1} = \mathcal{D} \left[ \frac{1}{2l+1} \mathcal{D}^{l+1}(x^2 - 1)^{l+1} \right] = \frac{1}{2(l+1)} \mathcal{D}^2 \left[ \frac{1}{2l+1} \mathcal{D}^{l+1}(x^2 - 1)^{l+1} \right]
\]
\[
= \frac{1}{2(l+1)} \frac{1}{2l+1} \mathcal{D}^2 \left[ (x^2 - 1)\mathcal{D}^{l}(x^2 - 1)^{l} + 2lx\mathcal{D}^{l-1}(x^2 - 1)^{l} + l(l-1)\mathcal{D}^{l-2}(x^2 - 1)^{l} \right]
\]
\[
= \frac{1}{2(l+1)} \left[ \mathcal{D}((x^2 - 1)\mathcal{D}P_l) + (2P_l + 2xP_l) + (2lx\mathcal{D}P_l + 4lP_l) + l(l-1)P_l \right].
\]
From Legendre’s equation, the first term is $l(l+1)P_l$. Gathering terms in $x \mathcal{D}P_l$ and $P_l$,
\[
\mathcal{D}P_{l+1} = \frac{1}{2(l+1)} \left[ 2(l+1)^2 P_l + 2(l+1)x \mathcal{D}P_l \right].
\]
or, finally, the desired result,
\[
\mathcal{D}P_{l+1} = (l+1)P_l + x \mathcal{D}P_l.
\](3.29b)
It seems likely that there is an easier way to get here, but this works.

Here is another one:
\[
\mathcal{D}P_{l+1} = \mathcal{D} \left[ \frac{1}{2l+1} \mathcal{D}^{l+1}(x^2 - 1)^{l+1} \right] = \frac{1}{2(l+1)} \frac{1}{2l+1} \mathcal{D}^{l+1} \left[ 2(x^2 - 1)^{l+1} \right]
\]
\[
= \frac{1}{2(l+1)} \frac{1}{2l+1} \mathcal{D}^{l+1} \left[ 4l(l+1)x(x^2 - 1)^{l-1} + 2(l+1)(x^2 - 1)^{l} \right]
\]
\[
= \frac{1}{2l+1} \mathcal{D}^{l+1} \left[ 2l(x^2 - 1)^{l} + (x^2 - 1)^{l} \right]
\]
\[
= \frac{1}{2l+1} \mathcal{D}^{l+1} \left[ 2l(x^2 - 1)^{l} + (2l+1)(x^2 - 1)^{l} \right] = \mathcal{D}P_{l-1} + (2l+1)P_l.
\]
From beginning to end, this says
\[
\mathcal{D}P_{l+1} - \mathcal{D}P_{l-1} = (2l+1)P_l.
\](3.28)