1. Debye model

(a) In a more realistic model of a crystal the modes of ionic oscillations do not have the exact acoustic spectrum $\omega = c_s k$ assumed in the Debye model of the low-\(T\) specific heat of a solid. Rather the modes are found to have a spectrum

$$\omega = \omega_0 \sin\left(c_s k / \omega_0\right),$$

where $\omega_0$ is a constant. You may assume no modes exist for $k > k_0 \equiv \pi \omega_0 / (2 c_s)$. The acoustic spectrum is then valid only for long wavelengths (small $k$). Redo the Debye calculation and derive the expression for the total internal energy density $u$ of the system,

$$u = \frac{1}{2\pi^2} \int_0^{k_0} dk k^2 \frac{\hbar \omega_0 \sin\left(c_s k / \omega_0\right)}{e^{c_s k / \omega_0} - 1}$$

(b) Verify that the Debye result $c_V \sim T^3$ is recovered approximately below a temperature $T_D$ (the “Debye temperature”), and specify what this temperature is in terms of the parameters $c_s$ and $\omega_0$ describing the energy spectrum of the ionic vibrations. What is the high temperature limit, $c_V(T \gg T_D)$?

(c) Using an analog of the Debye model described in the text, find the specific heat per unit area $c_{2D} \equiv du/dT$ of a two-dimensional solid. (Hint: the only thing which will be essentially different will be the dimensionless integral which occurs in the expression for the free energy, which you will need to evaluate with Maple or look up in an integral table.)
In the standard derivation of the Debye specific heat, it is assumed the sum over modes may be replaced by an integral. Make an order of magnitude estimate for the size of a cube of material at room temperature below which this approximation is no longer valid, i.e. the discreteness of the energy spectrum should become important. Assume the velocity of sound is size-independent, \( c_s = 1 \times 10^5 \text{m/s} \).

2. **Maple problems.** Late penalties waived for these problems. Please turn in Maple worksheet, i.e. both code and results.

   (a) Plot \( \sin(x)/x \) from 0 to 1. Label axes.
   (b) Plot \( \sin(xy) \) over (0..1),(0..1). Label axes.
   (c) Find the roots of \( x^3 - 13x + 12 \).
   (d) Find a root of \( e^{\cos x} = \log (2 + \sin x) \) in the range (0..2).
   (e) Differentiate the function \( \cos(\log x) \).
   (f) Integrate \( 1/(x^2 + a^2)^{3/2} \)
       i. indefinite integral
       ii. from 0. to 1.
   (g) Plot the result from part 1a) above as a function of \( T/T_D \).