1. Hydrogen wave functions.

(a) Plot and label the first six (i.e. \( n=1,2,3 \), all allowed values of \( \ell \)) radial functions \( R_{n\ell}(r) \), as functions of \( r/a_0 \). These functions are found to be

\[
R_{n\ell}(x) = \sqrt{\left( \frac{2}{n a_0} \right)^2 \frac{(n - \ell - 1)!}{2n[(n + \ell)!]^2}} e^{-x^2/2} x^\ell L_{n-\ell-1}^{2\ell+1}(x)
\]

where \( L_{n\ell} \) are associated Laguerre polynomials and \( x \equiv 2r/(na_0) \).

- (Option 1.) You may use Maple to calculate and plot these functions. In this case note that the built-in functions \( L(n, a, x) \) Maple calls generalized Laguerre polynomials are normalized differently from the associated Laguerre polynomials. To generate associated Laguerre polynomials, type:

\[
\text{Laguerre} := (q, xi) \rightarrow \text{expand}(\exp(xi)\cdot\text{diff}(xi^q\exp(-xi), xi\cdot q))
\]

This is the def. of Laguerre polynomials, \( L_q(\xi) = e^{\xi}(d/d\xi)^q(e^{-\xi}\xi^q) \).

Now type

\[
\text{ALaguerre} := (q, p, xi) \rightarrow (-1)^p \cdot \text{diff}(\text{Laguerre}(q, xi), xi\cdot p)
\]

to define the associated Laguerre polynomials, \( L_{q-p}^p(\xi) = (-1)^p(d/d\xi)^pL_q(\xi) \).

To get the required \( L_{n-\ell-1}^{2\ell+1} \) occurring in Eq. (1), we need \( p = 2\ell + 1 \) and \( q = n + \ell \). So define

\[
\text{Lnl} := (n, l, xi) \rightarrow \text{ALaguerre}(n+1, 2l+1, xi)
\]

Then you should get, e.g.

\[
\text{Lnl}(3, 0, xi);
\]

\[18 - 18\xi + 3\xi^2\]

Next define a radial function \( R_{n\ell} \) using the substitute command \( \text{subs} \):

\[
\text{Rnl} := (n, l, xi) \rightarrow \text{subs}(xi=2*r/n, \text{expr})
\]

where \( \text{expr} \) is your expression for the form of the \( R_{n\ell} \). This is necessary because if you try to define the Laguerre polynomial \( \text{Laguerre}(q, 2r/n) \), Maple tries to differentiate with respect to the variable \( 2 \cdot r/n \) and can’t, whereas \( \text{subs} \) substitutes \( 2 \cdot r/n \) after differentiation.

Plot your 6 \( R_{n\ell} \) vs. \( r/a_0 \) on a scale of \( r/a_0 = 0..20 \) and \( R = -0.2..0.6 \).
• (Option 2.) If you want to avoid Maple, derive the 6 $R_{n\ell}$ from the
normalization condition and the recursion relation discussed in class.
Sketch the curves for $R_{n\ell}$ vs. $r/a_0$, paying particular attention to the
values at $r = 0$, as well as the nodes (values of $r/a_0$ where $R_{n\ell} = 0$).

**N.B.** Do not compare with Fig. 4.4 in Griffiths, as it contains a mistake!
2 pts. extra credit for finding it...

(b) Plot the radial probability density $P(r)$ for finding the electron a distance
$r$ away from the nucleus for the three cases $n, \ell = (1,0)$, $(2,0)$, and $(3,0)$.
Hint: the radial probability density is not $R_{n\ell}^2$! Maple or sketch. Comment
on the number of nodes in the wavefunction and its dependence on $n$; also
on the most likely distance of the electron from the nucleus, and how it
depends on $n$.

(c) Suppose an electron is in an angular state $Y_{1,1} + Y_{1,-1}$. Plot the angular
probability distribution $P(\theta, \phi)$ by using

```maple
with(plots):
sphereplot(P(theta,phi),phi=0..2*Pi,theta=0..Pi,grid=[51,51],
     scaling=CONSTRAINED);
```

Sketch also ok. Specify what your plot means, but don’t worry about
normalization.

2. **Harmonic Oscillator.** Find the matrix elements $< n | x | n' >$ and $< n | p | n' >$
in the basis consisting of stationary states of the simple harmonic oscillator.
Construct the corresponding (infinite) matrices $\hat{X}$ and $\hat{P}$, and show that $H = \hat{P}^2/2m + (1/2)m\omega^2\hat{X}^2$ is diagonal in this basis. Give the diagonal elements.