1. Linearity. Given the following operators, which are linear?

(a) $\psi \rightarrow x^3 \psi$
(b) $\psi \rightarrow x \frac{d}{dx} \psi$
(c) $\psi \rightarrow \lambda \psi^*$
(d) $\psi \rightarrow e^\psi$
(e) $\psi \rightarrow \frac{d}{dx} \psi + \text{const.}$
(f) $\psi \rightarrow \int_0^x dx' \psi(x')$

2. Commutators. Calculate the commutators ($L_{\pm}$ are the SHO ladder operators defined in class):

(a) $[x^3, x \frac{d}{dx}]$
(b) $[\nabla^2, \frac{1}{r}]$
(c) $[x^2, \frac{d}{ds}]$
(d) $[3, x^2]$
(e) $[3, \frac{d}{dx}]$
(f) $[L_+, L_-]$
(g) $[L_+, L_+]$
(h) $[L_+, x]$
(i) $[L_+, p]$

3. Complex numbers. Simplify:

(a) $|3 + 2i|^2$
(b) $\text{Re} \sqrt{x^2 - 2ixy}$
(c) $\text{Im} \ e^{\omega t/\hbar}$
(d) $\text{Re} \frac{z^2 \lambda - 2iz}{z^2 - 2iz}$
4. Fourier transform. For these problems assume the asymmetric normalization of the Fourier transform:

\[ f(x) = \int g(k)e^{ikx} \]
\[ g(k) = \frac{1}{2\pi} f(x)e^{-ikx} \]

Try to do each of these analytically, then check with Maple. Hint: some may be easier if you use contour integration.

(a) \( f(x) = \) “window” function of width \( a \),

\[ f(x) = \begin{cases} \frac{A}{2} & |x| < a/2 \\ 0 & |x| > a/2 \end{cases} \]

\( g(k) =? \)

(b) \( f(x) = \) Gaussian \( \exp(-x^2/(2\sigma^2)) \), \( g(k) =? \)

(c) \( f(x) = A\exp(-ax|x|) \), \( g(k) =? \)

(d) Lorentzian \( f(x) = \frac{A}{(x^2+a^2)} \), \( g(k) =? \)

5. Projecting out amplitudes. Determine the amplitude desired in each problem:

(a) Expand window function above in 1D SHO eigenfunctions. What is the amplitude of the 1st excited state?

(b) Expand the sawtooth function

\[ f(x) = \begin{cases} x + a/2 & -a/2 < x < 0 \\ x & 0 < x < a/2 \\ 0 & \text{otherwise} \end{cases} \]

in terms of eigenstates of the infinite square well between \((-a/2, a/2)\). Determine the amplitude of the 1st and 2nd excited states.

6. Delta functions. Evaluate:

(a) \( \int_{-3}^{3} dx \delta(x - 2) \exp(-\frac{1}{2}x^2) \)

(b) \( \int_{-3}^{3} dx \delta(x - 4) \exp(-\frac{1}{2}x^2) \)

(c) \( \int d^3 r e^{ik\cdot r} e^{-i k\cdot r} \)

(d) \( \int d^3 r e^{ia k\cdot r} e^{-ib k\cdot r} \)

7. Inner products.
(a) Simplify \((\alpha A + \beta B, \gamma C + \delta D)\), where Greek letters are complex constants and Roman letters are operators.
(b) Evaluate \((\psi_0, x^2\psi_2)\) for SHO wavefunctions.
(c) Evaluate \((\psi_1, x^2\psi_2)\) for SHO wavefunctions.

8. **Hermitian operators.** Let \(x\) and \(p\) be the usual operators for position and momentum. State whether each of the following operators is self-adjoint (Hermitian), anti-self-adjoint (antiHermitian, \(O^\dagger = -O\)), unitary \((O^\dagger O = 1)\), or if none of the above, what the adjoint is. Hint: remember \((AB)^\dagger = B^\dagger A^\dagger\).

(a) \(xxp\)
(b) \(xpx\)
(c) \(xpp + ppx\)
(d) \(\frac{d^2}{dx^2}\)
(e) \(\frac{d}{dx}\)
(f) \(e^p\)

9. **More operators.** Two operators \(A\) and \(B\) satisfy \(A = B^\dagger B + 3\) and \(A = BB^\dagger + 1\).

(a) show \(A\) is self-adjoint
(b) find \([B^\dagger, B]\)
(c) find \([A, B]\)
(d) Suppose \(\psi\) is an eigenfunction of \(A\) with eigenvalue \(\alpha\). Show that if \(B\psi \neq 0\), then \(B\psi\) is an eigenfunction of \(A\), and find the eigenvalue.

10. **Gram-Schmidt.** The Legendre polynomials \(P_\ell(x)\) are a set of real polynomials orthogonal to each other on the interval \(-1 < x < 1:\)

\[
\int_{-1}^{1} dx P_\ell(x) P_{\ell'}(x) = \delta_{\ell,\ell'}.
\]

The polynomial \(P_\ell\) is of order \(\ell\), i.e. the highest power of \(x\) is \(x^\ell\). Starting with the set of functions \(\psi_\ell(x) = x^\ell\), use the Gram-Schmidt procedure to construct \(P_0(x)\), \(P_1(x)\), and \(P_2(x)\).