2 Quantization of Normal Modes

2.1 Wave equation

Want to review modes of electromagnetic radiation in cavity. Start with Maxwell’s equations in free space (SI units)

\[ \nabla \cdot \vec{E} = 0 \]  (1)
\[ \nabla \cdot \vec{B} = 0 \]  (2)
\[ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \]  (3)
\[ \nabla \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = 0 \]  (4)

where \( \nabla \) is gradient operator

\[ \nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \]  (5)

Will also need subsidiary condition, \textit{charge conservation}:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 \]  (6)

and Lorentz force:

\[ \vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \]  (7)

Now take curl of Eq. (3), and use identity \( \nabla \times (\nabla \times \vec{E}) = \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \):

\[ 0 = \nabla \times (\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t}) \]  (8)
\[= -\nabla^2 E + \frac{\partial}{\partial t} \nabla \times B \]  
(9)

\[\mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \]  
(10)

\[= -\nabla^2 E + \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} \]  
(11)

i.e. a wave equation for each component of the electric field \( E \) for waves travelling with speed \( c \equiv \sqrt{\mu_0 \varepsilon_0} \). Why is it a wave eqn.? Consider soln. \( E = E_0 f(x - ct) \), with \( f \) any function. Check! Compare field “profile” \( f(x) \) at time \( t = 0 \) with profile time \( t \) later: simply translated to right distance \( ct \Longrightarrow “pulse” \ f(x) \) moving to right with velocity \( c \) (vel. of light!)

Check that

1. \( E \perp \hat{i} \) (direction of propagation)

2. \( B \) field \( \perp \hat{i}, E \)

### 2.2 Normal modes of cavity

Example: transverse waves on string.

Displacement of string \( \delta(x, t) \), obeys

wave equation \[ \frac{\partial^2 \delta}{\partial t^2} = c_s^2 \frac{\partial^2 \delta}{\partial x^2} \]  
(12)
where $c_s \equiv$ velocity of sound. Boundary conditions: suppose

$$\delta(x = 0, t) = 0,$$  \hspace{1cm} (13)

$$\delta(x = L, t) = 0$$ \hspace{1cm} (14)

Normal modes of vibration: solutions of form

$$\delta = f(x) \cos(\omega t + \phi)$$ \hspace{1cm} (15)

consistent with boundary conditions. Note these are standing waves. Substitute in (12)–solution only if

$$-\omega^2 f(x) = c_s^2 \frac{d^2 f}{dx^2}.$$ \hspace{1cm} (16)

(16) called “Eigenvalue” problem (German: proper value) since eqn has solutions consistent with B.C. only for special values (eigenvalues) $\omega$.

General soln. to (16):

$$f(x) = A \sin \frac{\omega x}{c_s} + B \cos \frac{\omega x}{c_s}$$ \hspace{1cm} (17)

B.C. $\delta(x = 0) = 0 \implies B = 0$, $\delta(x = L) = 0 \implies A = 0$ or

$$\omega = n\pi c_s / L, \quad n = 1, 2, 3...$$ \hspace{1cm} (18)

N.B. This purely classical result has connection with quantization as used in SHO problem.
Modes in cavity: wave eqn. for electric field. Assume walls perfect conductors, so B.C. are $E_\parallel = 0$, $B_\perp = 0$ at walls. Search for solns of Maxwell eqns. of form

$$ E = \cos(\omega t + \phi)f(r)$$

consistent with these B.C.

### 2.3 Periodic boundary conditions

We can find solutions with fixed (standing wave) boundary conditions, but since it is sometimes convenient to use other B.C., let’s look at periodic B.C., i.e. assume cavity is periodically repeated with period $L$ along $x, y, z$ — $x, y, z$ is same pt. as $x + L, y, z$, etc. This is ok since the shape of cavity, exact pos. of walls can’t matter if we are calculating something which depends only on the density of modes in freq. range. such that $L \gg$ sizes of interest.

Consider solutions

$$ E = \text{Re} \, E_0 e^{i(kr - \omega t)}$$

(20)
\[ |E_0| \cos(k \cdot r - \omega t) \]  

(Eq. 21)

\[ E_0 = \text{const.} \quad \phi = \text{const} \]

Re=“real part of”

Remarks:

- Plug into wave eqn. \( \implies \omega^2 = k^2 c^2 \)

- \( \nabla \cdot E = 0 \implies k \cdot E_0 = 0 \) transverse wave

\[ k \cdot E_0 = 0 \]

- Periodic B.C. \( \implies \)

\[ k_x L = 2\pi n_x, \quad n_x = 0, \pm 1, \pm 2... \]

\[ k_y L = 2\pi n_y, \quad n_y = 0, \pm 1, \pm 2... \]

\[ k_z L = 2\pi n_z, \quad n_z = 0, \pm 1, \pm 2... \]  

(Eq. 22)

- Note allowed \( k \)'s separated by 2× larger interval compared to fixed b.c., but \( n_{x,y,z} \) can be \( \pm \implies \) same # per \( d\nu \).

- note that for, e.g., \( n_x = \pm 1, \) + and - give independent solutions – check! This was not true for fixed B.C.

Allowed frequencies are therefore
\[ \omega = kc = \frac{2\pi c}{L} (n_x^2 + n_y^2 + n_z^2)^{1/2}. \]  

Maxwell’s eqns linear—lin. comb. of solns. is soln. General:

\[ \mathbf{E} = \text{Re} \sum_{p=1,2} \sum_{n_x,n_y,n_z} \mathbf{E}_0(n,p) e^{i(k \cdot r) - \omega t} \]  

here

\( p \) is polarization of wave: 2 lin. ind. directions of \( \mathbf{E} \) for given \( \mathbf{k} \)

\[ \mathbf{n} = n_x, n_y, n_z \]

\[ \mathbf{k} = 2\pi \mathbf{n}/L \]

Exercise: derive form for \( \mathbf{B} \) from Eq. (24) and James Clerk’s equations...

### 2.4 Counting modes

Need # of solutions \( \sim e^{i(k \cdot r - \omega t)} \) with freqs in range \( \nu \) to \( \nu + \delta \nu \). Look at “k-space”:

![Diagram](attachment:image.png)
• Allowed $k$ form cubic lattice in $k$-space, spacing $2\pi/L$

• “volume” occupied by each allowed $k$ is $(2\pi/L)^3$

• volume in $k$-space between $k$ and $k + \delta k$ is $4\pi k^2 \delta k$

• therefore number of $k$’s in this shell is

$$\delta N = 4\pi k^2 \delta k \cdot (L/2\pi)^3$$  \hspace{1cm} (25)

Now relate number of $k$’s in $(k, k + \delta k)$ to number of allowed solutions with $\nu$’s in $(\nu, \nu + \delta \nu)$ (recall $\omega = 2\pi \nu = kc$!):

$$\delta N = 2 \cdot 4\pi \cdot \frac{L^3}{8\pi^3} \cdot \left( \frac{2\pi}{c} \right)^3 \nu^2 \delta \nu$$

2 polarizations $\theta, \phi$ integration $k = 2\pi \nu/c$

$$= 8\pi L^3 \nu^2 \delta \nu / c$$ \hspace{1cm} (26)

2.5 Planck Law

Planck idea (1900): each mode of radiation in cavity acts like SHO — suppose it contributes mean energy $h\nu/(\exp^{h\nu/kT} - 1)$. Mean energy in $(\nu, \nu + \delta \nu)$ is then

$$\delta E = \frac{8\pi L^3 h \nu^3 \delta \nu}{c^3 (e^{h\nu/kT} - 1)}$$ \hspace{1cm} (27)

Energy per unit volume and bandwidth:

$$u_\nu \equiv \frac{1}{L^3} \frac{\delta E}{\delta \nu} = \frac{8\pi h \nu^3}{c^3 (e^{h\nu/kT} - 1)}$$ \hspace{1cm} (28)
(Aside: at high temperatures, $\hbar \nu / kT \ll 1$, expand exponential to find $u_{\nu} = 8\pi kT \nu^2 / c^3$, Rayleigh-Jeans law. Note there’s no $\hbar$–classical result.)

Net energy density now

$$u = \int_0^\infty u_\nu d\nu = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3 d\nu}{(e^{\hbar \nu / kT} - 1)}$$  \hspace{1cm} (29)

Make integral dimensionless, put $\alpha = \hbar \nu / kT$, $d\nu = kT d\alpha / \hbar$:

$$u = \frac{8\pi h}{c^3} \left(\frac{kT}{\hbar}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1} \frac{\pi^4}{15}$$  \hspace{1cm} (30)

So Planck also recovered:

$$u = aT^4 \quad \text{Stephan-Boltzmann law}$$  \hspace{1cm} (31)

and determined

$$a = \frac{8\pi^5 k^4}{15 (hc)^3}$$  \hspace{1cm} (32)

But if had let $\hbar \to 0$ as originally intended, → nonsense! Although made no sense classically, P. brave enough to allow $\hbar$ to stay. Stephan’s const. pretty well known expt’lly in 1900 ⇒ Planck determined $\hbar = 6.55 \times 10^{-27}$ erg – sec, close to modern $\hbar = 6.625 \times 10^{-27}$ erg – sec!