Euler-Lagrange Equations for charged particle in a field

The Lagrangian is
\[ L = \frac{1}{2} m \dot{r}^2 + q(A \cdot \dot{r} - \phi) \]

Euler Lagrange Equations are
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \]
so calculate left and right hand sides separately:
\[ \frac{\partial L}{\partial r} = q \frac{\partial}{\partial r} (A \cdot \dot{r}) - q \frac{\partial \phi}{\partial r} \]
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = m \ddot{r} + q \frac{d}{dt} A \]

Now recall \( A \) is the vector potential evaluated at the position of the particle \( r \) at time \( t \). The particle is following a trajectory \( r(t) \), so \( A = A(r(t), t) \). The total time derivative thus gives two terms,
\[ \frac{d}{dt} A = \frac{\partial A}{\partial t} + \frac{\partial r}{\partial t} \cdot \frac{\partial}{\partial r} A \]
or, to be completely clear, for a given component \( A_i \),
\[ \frac{d}{dt} A_i = \frac{\partial A_i}{\partial t} + \frac{\partial r_j}{\partial t} \cdot \frac{\partial}{\partial r_j} A_i \]
where a sum over repeated indices is implied. So, putting the Euler-Lagrange equation together for index \( i \) gives
\[ m \ddot{r}_i = -q \frac{\partial}{\partial r_i} \phi - q \frac{\partial A_i}{\partial t} + q \left( \frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j} \right) \dot{r}_j \]
\[ = q (E + \dot{r} \times B)_i \]
which is the \( ith \) component of the Lorentz force. To convince yourself that the last term in parentheses really turns into the \( \dot{r} \times B \) term, evaluate
\[ (\dot{r} \times B) = (\dot{r} \times (\nabla \times A))_i = \epsilon_{ijk} \dot{r}_j (\nabla \times A)_k = \epsilon_{ijk} \dot{r}_j \epsilon_{k\ell m} \frac{\partial}{\partial r_\ell} A_m \]
\[ = \left( \delta_{i\ell} \delta_{km} - \delta_{i\ell} \delta_{km} \right) \dot{r}_j \frac{\partial}{\partial r_\ell} A_m \]
\[ = \dot{r}_j \frac{\partial}{\partial r_i} A_j - \dot{r}_j \frac{\partial}{\partial r_j} A_i = \left( \frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j} \right) \dot{r}_j \]
\[ QED \]
(I used the cyclic property of the Levi-Civita symbol \( \epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} \), and the identity \( \epsilon_{ijk} \epsilon_{ilm} = \delta_{j\ell} \delta_{km} - \delta_{jm} \delta_{k\ell} \).