B1. (a) (4 points) Calculate the expectation value $\langle \frac{r^2}{2} \rangle$ for the groundstate 1s ($n = 1$, $\ell = 0$) electron in hydrogen, where the wave function is given by

$$\Psi_{10} = \left( \frac{1}{\pi a_0^2} \right)^{1/2} \exp\left( -\frac{r}{a_0} \right),$$

and the Bohr radius $a_0 = 0.529 \text{ Å}$. Show all of your work for full credit.

(b) (2 points) What is the expectation value of $L^2$ in the eigenstate given in part (a)?

(c) (4 points) What is the expectation value of $L^2$ for the hydrogen atom eigenstate with wavefunction

$$\Psi = \frac{1}{4} \left( \frac{1}{2\pi a_0^2} \right)^{1/2} (\cos \theta) \left( \frac{r}{a_0} \right) \exp\left( -\frac{r}{2a_0} \right)?$$
B2. Three identical radio broadcast towers lie along the y-axis and are a distance \( d = 10 \text{ m} \) apart (the middle tower is at \( y = 0 \)). All three towers broadcast at a frequency of 100 MHz with equal intensities and in phase. There is a road parallel to the y-axis a distance \( L = 10 \text{ km} \) from the radio towers as shown in the figure. An observer walking along the road with a radio tuned to 100 MHz receives an intensity \( I_o \) at \( y = 0 \) (i.e. \( I(\theta = 0) = I_o \)). Use the small angle approximation when answering the questions since \( L \gg d \).

(a) (3 points) If the observer starts at \( y = 0 \) and begins walking in the positive y direction, at what distance \( y \) (in km) will the intensity of the radio signal be zero due to interference between the three towers (i.e. at what distance \( y \) does the first maximal destructive interference occur)?

(b) (2 points) The intensity of the radio signal received by the observer on the road at \( y = 1.5 \text{ km} \) may be written as \( I(y = 1.5 \text{ km}) = \alpha I_o \). What is \( \alpha \)?

(c) (2 points) The intensity of the radio signal received by the observer on the road at \( y = 3.0 \text{ km} \) may be written as \( I(y = 3.0 \text{ km}) = \beta I_o \). What is \( \beta \)?

(d) (3 points) Sketch the intensity of the radio signal as a function of the distance along the road. In other words, plot \( I(y)I_o \) from \( y = 0 \) to \( y = 4 \text{ km} \).
B3. A pendulum is constructed by attaching a mass \( m \) to an extensionless string of length \( l \). The upper end of the string is then connected to the uppermost point on a vertical disk of radius \( R \) \((R < l/\pi)\) as shown in the figure.

(a) (2 points) Find the \( x \) and \( y \) coordinates of the mass \( m \) in terms of the angle \( \theta \). (Use the coordinate axes shown in the figure).

(b) (3 points) Using \( \theta \) as the generalized coordinate, find the Lagrangian for this system and hence provide the equation of motion in the form (i.e. you will need to provide \( \alpha \), \( \beta \), and \( \gamma \), if this general form is the correct one):

\[
\alpha \ddot{\theta} - \beta \dot{\theta}^2 - \gamma \cos \theta = 0
\]

(c) (3 points) Let \( \epsilon = \theta - \theta_0 \) \((\epsilon \ll \theta_0)\). What is the frequency of small oscillations of the mass \( m \) about the angle \( \theta = \theta_0 \)?

(d) (2 points) Find the \( \theta_0 \) for which the angular motion extends equally in either direction (i.e. \( \dot{\theta}_1 = \dot{\theta}_2 \)).