CCMS Summer 2007 Lecture Series

Fermi- and non-Fermi Liquids

Lecture 1: Basic Notions of the Many-Body Physics

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I. SECOND QUANTIZATION

A. Occupation number representation

Systems of many particles are conveniently described by the second quantization representation. Suppose that we have a systems of particles (bosons or fermions), each of which can be in one of the states $\psi_i, i = 1, 2, \ldots$ The many-body wavefunction can be written down in the occupation number representation, which specifies the number of particles occupying each of the quantum states. Dirac notation for the state vector

$$| \ldots, N_{i-1}, \underbrace{N_i}_{\text{# of particles in state } i}, N_{i+1}, \ldots \rangle.$$  

For bosons, $N_i = 0, 1, 2, \ldots$ For fermions, $N_i = 0, 1$.

Creation and annihilation operators are defined by their action on the state vector. For bosons,

$$a_i | \ldots, N_{i-1}, N_i, N_{i+1}, \ldots \rangle = \sqrt{N_i} | \ldots, N_{i-1}, N_i - 1, N_{i+1}, \ldots \rangle$$  
$$a_i^\dagger | \ldots, N_{i-1}, N_i, N_{i+1}, \ldots \rangle = \sqrt{N_i + 1} | \ldots, N_{i-1}, N_i + 1, N_{i+1}, \ldots \rangle.$$  

For fermions,

$$a_i | \ldots, N_{i-1}, N_i, N_{i+1}, \ldots \rangle = | \ldots, N_{i-1}, 0, N_{i+1}, \ldots \rangle \text{ for } N_i = 1$$  
$$0 \text{ for } N_i = 0$$  
$$a_i^\dagger | \ldots, N_{i-1}, N_i, N_{i+1}, \ldots \rangle = | \ldots, N_{i-1}, 1, N_{i+1}, \ldots \rangle \text{ for } N_i = 0$$  
$$0 \text{ for } N_i = 1$$

Commutation relations

$$[a_i^\dagger, a_j]_{\pm} \equiv a_i^\dagger a_j \pm a_j a_i^\dagger = \delta_{ij}$$  
$$[a_i, a_j]_{\pm} = [a_i^\dagger, a_j^\dagger]_{\pm} = 0.$$  

The ± sign refers to fermions (bosons).

B. $\Psi$-operators

A $\Psi$-operator is defined as
\[ \psi (\mathbf{r}) = \sum_i \psi_i (\mathbf{r}) a_i \] (1.1)

\[ \psi^\dagger (\mathbf{r}) = \sum_i \psi_i^* (\mathbf{r}) a_i^\dagger, \]

where \( \psi_i (\mathbf{r}) \) is the wave function, i.e., a solution of the Shroedinger equation, of state \( i \).

As we will be dealing with non-relativistic Shroedinger equation, it is convenient to choose \( \psi_i (\mathbf{r}) \) to be independent of the spin, and to assign an additional quantum number (spin projection) to the \( a_\cdot \) operators. Then the definition of the \( \hat{\psi} (\mathbf{r}) \) operator changes to

\[ \hat{\psi}_\alpha (\mathbf{r}) = \sum_i \psi_i (\mathbf{r}) a_{i\alpha} \]

\[ \hat{\psi}_\alpha^\dagger (\mathbf{r}) = \sum_i \psi_i^* (\mathbf{r}) a_{i\alpha}^\dagger, \]

Now the sum runs only over the orbital states. In most cases, the orbital states \( \psi_i (\mathbf{r}) \) form a complete basis. In this case,

\[ \left[ \hat{\psi}_\alpha^\dagger (\mathbf{r}), \hat{\psi}_\beta^\dagger (\mathbf{r}') \right]_{\pm} = \sum_{ij} \psi_i^* (\mathbf{r}) \psi_j (\mathbf{r}') \left[ a_{i\alpha}, a_{j\beta} \right]_{\pm} = \sum_{ij} \psi_i^* (\mathbf{r}) \psi_j (\mathbf{r}') \delta_{ij} \delta_{\alpha\beta} = \delta_{\alpha\beta} \sum_i \psi_i^* (\mathbf{r}) \psi_i (\mathbf{r}') \]

\[ = \delta_{\alpha\beta} \delta (\mathbf{r} - \mathbf{r'}). \]

\( \hat{\psi}_\alpha^\dagger \)-operator creates a particle at point \( \mathbf{r} \) with spin \( \alpha \) regardless of its orbital quantum state.

**Example 1** \( \hat{\psi} \) -operator for free particles

\[ \hat{\psi}_\alpha (\mathbf{r}) = \frac{1}{\sqrt{\text{vol}}} \sum_{\mathbf{p}} e^{i\mathbf{p}\cdot \mathbf{r}} a_{\mathbf{p}\alpha} \] (1.2)

\[ \hat{\psi}_\alpha^\dagger (\mathbf{r}) = \frac{1}{\sqrt{\text{vol}}} \sum_{\mathbf{p}} e^{-i\mathbf{p}\cdot \mathbf{r}} a_{\mathbf{p}\alpha}^\dagger, \]

where "vol" denotes the system volume and \( \mathbf{p} \) is the wavevector.

**Example 2** \( \hat{\psi} \) -operator for free particles in a magnetic field (in the Landau basis)

\[ \hat{\psi}_\alpha (\mathbf{r}) = \sum_n \sum_{p_z} \frac{1}{\sqrt{L_z}} \sum_{p_x} \psi_{np_z} (\mathbf{r}) a_{\mathbf{p}\alpha}, \]

where \( n \) is the Landau level index, \( p_z \) is the wavevector along the field (\( z \)-direction), \( p_x \) is auxiliary quantum number specifying the degeneracy of the quantum state, and

\[ \psi_{np_z} (\mathbf{r}) = e^{i p_z z} e^{i p_x x} \frac{1}{\pi^{1/4} \sqrt{2^n n! \ell_B}} \exp \left( -\frac{(y + p_x \ell_B^2)^2}{2 \ell_B^2} \right) H_n \left( \frac{y + p_x \ell_B^2}{\ell_B} \right) \]

with \( \ell_B = \sqrt{\hbar c/eB} \) being the magnetic length and \( H_n (x) \) being the Hermite polynomial of order \( n \) [1].
C. Hamiltonians in the second-quantized form

Example 3 A Hamiltonian for a system of particles subject to an external potential $U(r)$ and interacting with each other via a two-body, spin-independent potential $V(r)$ is

$$H = \sum_{\alpha} \int d^d r \left[ \frac{\hbar^2}{2m} \nabla^2 \right] \hat{\Psi}_\alpha (r) + U(r) \hat{\Psi}_\alpha^\dagger (r) \hat{\Psi}_\alpha (r)$$

$$+ \frac{1}{2} \sum_{\alpha \beta} \int d^d r \int d^d r' V(r - r') \hat{\Psi}^\dagger_\alpha (r) \hat{\Psi}^\dagger_\beta (r') \hat{\Psi}_\beta (r') \hat{\Psi}_\alpha (r).$$

Notice that the form of the Hamiltonian does not depend on the statistics, which enters only through the commutation properties of the $\hat{\Psi}$-operators [2]. Substituting expansion (1.2) into (1.3) and introducing Fourier transforms $V(q) = \int d^d r e^{i q \cdot r} V(r)$, $U(q) = \int d^d r e^{i q \cdot r} U(r)$, we obtain the Hamiltonian in terms of $a$-operators

$$H = \sum_k \left[ \frac{\hbar^2 k^2}{2m} a^\dagger_k a_k + \frac{1}{vol} \sum_q U(q) a^\dagger_{k+q} a_k \right]$$

$$+ \frac{1}{2} \frac{1}{vol} \sum_{\alpha \beta} \sum_{k,p,q} V(q) a^\dagger_{k+q} a^\dagger_p a_{p-q\beta} a_k.$$

II. GREEN’S FUNCTIONS

A. Classical Physics

In Classical Physics, a Green’s function of a linear operator is defined as a response of the classical equation of motion to a perturbation localized in space and time

$$\hat{L} G = \delta (r - r') \delta (t - t'),$$

where $\hat{L}$ is an operator.

Example 4 Green’s function of the diffusion equation satisfies

$$\left( \frac{\partial}{\partial t} - D \nabla^2 \right) G(r',r''t) = \delta (r - r') \delta (t - t').$$

(2.1)
**B. Time-ordered Green’s functions in Quantum Physics**

In Quantum Physics, the Green’s functions are introduced as expectation values of combinations of the \( \hat{\Psi} \)-operators. For example, the single-particle time-ordered (causal) Green’s function is defined as

\[
G_{\alpha\beta}(rt, r't') = -i\langle \hat{T} \hat{\Psi}_\alpha(rt) \hat{\Psi}_\beta^\dagger(r't') \rangle.
\]

Here \( \langle \ldots \rangle \) denotes averaging over the ground state, \( \hat{\Psi}_\alpha(rt) \) is the \( \hat{\Psi} \)-operator in the Heisenberg representation, and \( \hat{T} \) is the time-ordering operator (not to be confused with the temperature!), defined as

\[
\hat{T} f(t_1) g(t_2) = \begin{cases} f(t_1) g(t_2) & \text{for } t_1 > t_2 \\ -g(t_2) f(t_1) & \text{for } t_1 < t_2 \end{cases}
\]

where the first definition is for fermions (the minus sign is due the interchange of two fermions) and the second one is for bosons. The Heisenberg representation of an operator is defined as

\[
\hat{\Psi}_\alpha(rt) = e^{i\hat{H}t} \hat{\Psi}_\alpha(r) e^{-i\hat{H}t},
\]

\[
\hat{\Psi}_\alpha^\dagger(rt) = e^{i\hat{H}t} \hat{\Psi}_\alpha^\dagger(r) e^{-i\hat{H}t},
\]

where \( \hat{\Psi}_\alpha(r) \) and \( \hat{\Psi}_\alpha^\dagger(r) \) are the operators in the Schrödinger representation, defined by (1.1).

**Example 5** *Heisenberg representation for free particles*

\[
H = \sum_{k\alpha} \varepsilon_k a_{k\alpha}^\dagger a_{k\alpha}
\]
\[ \hat{\Psi}_\alpha (rt) = e^{iHt} \hat{\Psi}_\alpha (r) e^{-iHt} = \frac{1}{\sqrt{\text{vol}}} \sum_p e^{ip \cdot r} e^{it \sum_{k_\alpha} \xi_k a_{k_\alpha}^\dagger a_{k_\alpha}} a_{p_\alpha} e^{-it \sum_{k'_{\beta'}} \xi_{k'_{\beta'}} a_{k'_{\beta'}}^\dagger a_{k'_{\beta'}}} \]

(2.2)

\[ = \frac{1}{\sqrt{\text{vol}}} \sum_p e^{ip \cdot r} \left( 1 + it \sum_{k_\alpha} \xi_k a_{k_\alpha}^\dagger a_{k_\alpha} + \ldots \right) a_{p_\alpha} \left( 1 - it \sum_{k'_{\beta'}} \xi_{k'_{\beta'}} a_{k'_{\beta'}}^\dagger a_{k'_{\beta'}} + \ldots \right) \]

\[ = \frac{1}{\sqrt{\text{vol}}} \sum_p e^{ip \cdot r} (a_{p_\alpha} - it \xi_p a_{p_\alpha} + \ldots) = \frac{1}{\sqrt{\text{vol}}} \sum_p a_{p_\alpha} e^{i(p \cdot r - \xi_p t)} \]

Physically, the Green’s function is a probability amplitude that a particle created at space-time point \( r' t' \) disappears at space-time point \( r t \). Mathematically speaking, the Green’s function is a solution of the Schroedinger equation with a delta-function source term. Different types of Green’s function correspond to different ways of imposing the initial conditions in time for this equation.

In Statistical Physics, the role of the total energy is taken over by the free energy, defined for a particular thermodynamic ensemble. It is convenient to work with a system in which the chemical potential is fixed but the number of particles can change (grand canonical ensemble). The corresponding free energy at \( T = 0 \) is the total energy minus the Lagrange term \( \mu N \). In Quantum Statistics, we introduce a new Hamiltonian

\[ \hat{H}' = \hat{H} - \mu \hat{N} = \hat{H} - \mu \sum_{p_\alpha} a_{p_\alpha}^\dagger a_{p_\alpha}. \]

For free particles,

\[ \hat{H}' = \sum_{p_\alpha} \xi_p a_{p_\alpha}^\dagger a_{p_\alpha} \]

where \( \xi_p = \xi_p - \mu \) is the energy measured from the chemical potential. This choice is especially convenient for Fermi systems, where the energy above or below the filled Fermi sea is the only real energy we are concerned with. The Heisenberg operators simply change to

\[ \hat{\Psi}_\alpha (rt) = e^{i\hat{H}'t} \hat{\Psi}_\alpha (r) e^{-i\hat{H}'t}. \]

The Heisenberg form of a free \( \hat{\Psi} \) operator is obtained from (2.2) by replacing \( \xi_p \rightarrow \xi_p \):

\[ \hat{\Psi}_\alpha (rt) = \frac{1}{\sqrt{\text{vol}}} \sum_p a_{p_\alpha} e^{i(p \cdot r - \xi_p t)} \]

Heisenberg operators are used to construct the Green’s functions.

**Example 6** Time-ordered Green’s functions for a Fermi gas. In equilibrium, it does not matter when we choose the initial time, so we can set \( t' = 0 \). Also, if the system is translationally invariant, we can choose \( r' = 0 \). Then
\[ G_{\alpha\beta} (rt) = G_{\alpha\beta} (rt, 00) = -i\langle \hat{T} \hat{\Psi}_\alpha (rt) \hat{\Psi}_\beta^\dagger (00) \rangle = -i \hat{\Psi}_\alpha (rt) \hat{\Psi}_\beta^\dagger (00) \text{ for } t > 0 \\
\]
\[ = i \frac{1}{\text{vol}} \sum_{pp'} e^{i(p-r-x_p)t} (a_{p\alpha} a_{p'\beta}^\dagger) \]
\[ = i \frac{1}{\text{vol}} \sum_{pp'} e^{i(p-r-x_p)t} (a_{p'\beta} a_{p\alpha}^\dagger). \]

Using the commutation relations, we obtain

\[ \langle a_{p'\beta}^\dagger a_{p\alpha} \rangle = \delta_{pp'} \delta_{\alpha\beta} n_p \]
\[ \langle a_{p\alpha} a_{p'\beta}^\dagger \rangle = \delta_{pp'} \delta_{\alpha\beta} \left( 1 - \langle a_{p'\beta}^\dagger a_{p\alpha} \rangle \right) = \delta_{pp'} \delta_{\alpha\beta} (1 - n_p), \]

where \( n_p = 1, \) for \( p < p_F \)
\( = 0, \) for \( p > p_F \)

\( \theta(p_F - p) \) and \( 1 - n_p = \theta(p - p_F), \) where \( \theta = 1, \) for \( x > 0 \)
\( = 0, \) for \( x < 0 \)

Thus

\[ G_{\alpha\beta} (rt) = -i \frac{\delta_{\alpha\beta}}{\text{vol}} \sum_p e^{i(p-r-x_p)t} \theta(p - p_F), \quad t > 0 \]
\[ = i \frac{\delta_{\alpha\beta}}{\text{vol}} \sum_p e^{i(p-r-x_p)t} \theta(p_F - p), \quad t < 0 \]

It is more instructive to find a Fourier transform of the Green’s function

\[ G_{\alpha\beta} (\varepsilon, p) = \int dt \int d^4r e^{i(\varepsilon t - p \cdot r)} G_{\alpha\beta} (rt) \]
\[ = -i \theta(p - p_F) \int_0^\infty dt e^{i(\varepsilon - \xi_p)t} + i \theta(p_F - p) \int_0^\infty dt e^{i(\varepsilon - \xi_p)t} n_p. \]

To regularize the divergent integrals over time, we add infinitesimally small imaginary parts to the arguments of the exponential

\[ \int_0^\infty dt e^{i(\varepsilon - \xi_p)t} \rightarrow \int_0^\infty dt e^{i(\varepsilon - \xi_p + i\delta)t} = \frac{i}{\varepsilon - \xi_p + i\delta} \]
\[ \int_{-\infty}^0 dt e^{i(\varepsilon - \xi_p)t} n_p \rightarrow \int_{-\infty}^0 dt e^{i(\varepsilon - \xi_p - i\delta)t} n_p = -\frac{i}{\varepsilon - \xi_p - i\delta}, \]

where \( \delta > 0. \) Combining the result,

\[ G_{\alpha\beta} (\varepsilon, p) = \delta_{\alpha\beta} \left[ \frac{\theta(p - p_F)}{\varepsilon - \xi_p + i\delta} + \frac{\theta(p_F - p)}{\varepsilon - \xi_p - i\delta} \right]. \]

This expression can be written in a more compact form as

\[ G_{\alpha\beta} (\varepsilon, p) = \delta_{\alpha\beta} \frac{1}{\varepsilon - \xi_p + i\delta \text{sgn}(p - p_F)}. \]

(2.3)
Since we are interested in the properties of the Green’s function near its pole \( \varepsilon = \xi_p = p^2/2m - p_F^2/2m \), the sign of \( p - p_F \) coincides with that of \( \varepsilon \). Then Eq.(2.3) can be re-written as

\[
G_{\alpha\beta} (\varepsilon, p) = \delta_{\alpha\beta} \frac{1}{\varepsilon - \xi_p + i\delta sgn \varepsilon}.
\]  
(2.4)

Without the \( \delta \) term in the denominator, the Green’s function is obviously just a solution of the Shroedinger equation

\[
\hat{H} G = \varepsilon G - i
\]

The \( \delta \) term is very important as it brings in the information about the Fermi sea. Also, the similarity between the quantum Green’s functions and Green’s functions of the linear classical equations of motion exists only as long as we are considering free particles. Quantum Green’s functions are defined for interacting systems as well which formally correspond to a nonlinear Shroedinger equation.

**Example 7** Time-ordered Green’s function for phonons. Displacement operator for an elastic continuum is a linear superposition of plane waves

\[
\hat{u} (r, t) = \frac{1}{\sqrt{\text{vol}}} \sum_q \frac{1}{\sqrt{2\rho \omega_q}} \left( b_q e^{i(\mathbf{q} \cdot \mathbf{r} - \omega_q t)} + b_q^\dagger e^{-i(\mathbf{q} \cdot \mathbf{r} - \omega_q t)} \right),
\]

where \( \rho \) is the mass density of the material, \( b_q \) (\( b_q^\dagger \)) is the bosonic annihilation (creation) operator, and \( \omega_q = sq \) is the dispersion of the acoustic sound mode (\( s \) is the speed of sound). Electrons couple to deformations produced by phonons. A uniform deformation means displacement of the crystal as a whole, which obviously cannot have any physical consequences. Therefore, the energy of the electron-phonon interaction can contain only the divergence of \( \hat{u} \). The corresponding Hamiltonian is

\[
\hat{H}_{e-ph} = \Lambda \int d^d r \hat{\Psi}_\alpha^\dagger (r) \nabla \cdot \hat{u} \hat{\Psi}_\alpha (r) = i\Lambda \sum_{k,q} a_{k+\mathbf{q}a}^\dagger a_{k\mathbf{q}} \mathbf{q} \cdot \hat{u}_q \mathbf{q},
\]

where \( \Lambda \) is the “deformation potential constant”. As only the combination \( i\mathbf{q} \cdot \hat{u}_q \mathbf{q} \) occurs in the theory, it is convenient to define a new field

\[
\phi (r, t) = \frac{i}{\sqrt{\text{vol}}} \sum_q \sqrt{\omega_q} \left( b_q e^{i(\mathbf{q} \cdot \mathbf{r} - \omega_q t)} + b_q^\dagger e^{-i(\mathbf{q} \cdot \mathbf{r} - \omega_q t)} \right)
\]

which is used to build a Green’s function for phonons

\[
D (r, t, r', t') = -i \langle \hat{T} \phi (r, t) \phi (r', t') \rangle.
\]
A calculation similar to the case of a free Fermi gas yields

\[ D(\omega, q) = \frac{\omega^2_q}{\omega^2 - \omega^2 + i\delta}. \]

C. Relation of the Green’s function to the observable properties

Green’s function by itself describes a process that cannot happen: in non-relativistic quantum mechanics particles are not created and annihilated. However, different contractions of the Green’s function are directly related to observable properties.

An expectation value of a single-particle operator, i.e., an operator acting on coordinates of only one particle, is

\[ \langle f \rangle = \sum_\alpha \int d^d r \langle \hat{\Psi}_\alpha^\dagger (r) \hat{f}(r) \hat{\Psi}_\alpha (r) \rangle. \]

On the other hand, for \( t \to 0^- \) (meaning that \( t \) approaches 0 from below)

\[ G_{aa} (r_t, r'0) = \mp i \langle \hat{\Psi}_\alpha (r'0) \hat{\Psi}_\alpha^\dagger (r) \rangle \]

where the upper sign refers to bosons and the lower one to fermions. Therefore,

\[ \langle f \rangle = \pm i \sum_\alpha \int d^d r \left[ \hat{f}(r) (\mp i) \langle \hat{\Psi}_\alpha^\dagger (r0) \hat{\Psi}_\alpha^\dagger (r'0) \rangle \right] \bigg|_{r \to r'} \]

where the limit \( r' \to r \) is taken after \( \hat{f} \) acts on \( G_{aa} \). For example, the number density operator is

\[ \hat{n} = \sum_i \delta (r - r_i) \]

so that the single particle operator \( \hat{f} \) is just \( \delta (r - r_1) \). According to (2.5), the expectation value of \( \hat{n} \), i.e., the observable number density, is

\[ n \equiv \langle \hat{n} \rangle = \pm i \sum_\alpha \int d^d r \delta (r - r_1) G_{aa} (r_t, r'0) \big|_{t \to 0^-} = \pm i \sum_\alpha G_{aa} (r_1 t, r_1 0) \big|_{t \to 0^-}. \]

Relabelling \( r_1 \to r \), we obtain quite a useful relation

\[ n (r) = \pm i \sum_\alpha G_{aa} (r t, r 0) \big|_{t \to 0^-}. \]

For a translationally invariant system, it is convenient to express \( n \) via the Green’s function in the momentum space

\[ n = \pm \sum_\alpha \int \frac{d^d p}{(2\pi)^d} \int \frac{d\varepsilon}{2\pi} e^{-i\varepsilon t} G_{aa} (\varepsilon, p) \big|_{t \to 0^-}. \]
On the other hand,
\[
n = \sum_\alpha \int \frac{d^d p}{(2\pi)^d} n_{p \alpha},
\]
where \(n_{p \alpha}\) is the occupation number. Comparing the two last results, we find that
\[
n_{p \alpha} = \pm \int \frac{d\varepsilon}{2\pi} e^{-i\varepsilon t} G_{\alpha \alpha} (\varepsilon, p) |_{t \to -}. \tag{2.7}
\]
Notice that this relation is exact.

**Exercise 8** Show that Eq.(2.7) yields the Fermi function if one uses the Green’s function for a free Fermi gas, Eq.(2.3).

**Exercise 9** Show that the number current is expressed via the Green’s function as
\[
j (r) = \pm \frac{1}{m} \left( \nabla_r - \nabla_{r'} \right) G_{\alpha \alpha} (r, r') |_{t \to -, r' \to r}. \tag{2.8}
\]

## D. Retarded and advanced Green’s functions

Time-ordered Green’s functions are not analytic functions of frequency \(\varepsilon\) (cf. Eq.(2.4)). This poses some inconveniences in the applications. A more convenient type of Green’s functions are the retarded and advanced ones. They defined in the same way as retarded and advanced Green’s functions are defined in Classical Physics: the retarded Green’s function corresponds to perturbation that occurs at \(t = 0\). The advanced one is time-reversed: it corresponds to a perturbation that occurs at \(t = \infty\). In Quantum Physics, the retarded and advanced Green’s function are defined as

\[
G^R (r, r') = -i \theta (t - t') \langle \Psi_\alpha^\dagger (r) \Psi_\beta (r') \Psi_\beta^\dagger (r') \Psi_\alpha (r) \rangle
\]
\[
G^A (r, r') = i \theta (t' - t) \langle \Psi_\alpha^\dagger (r) \Psi_\beta (r') \Psi_\beta^\dagger (r') \Psi_\alpha (r) \rangle.
\]

The first sign is for bosons and the second one is for fermions. By construction, \(G^R = 0\) for \(t < t'\), i.e., the system responds to the perturbation applied at time \(t'\) only at later times. Similarly, \(G^A = 0\) for \(t > t'\).

In equilibrium, one can set \(t' = 0\) so that the arguments of the Green’s functions become \(G^R (r, r')\) and \(G^A (r, r')\).

**Exercise 10** Show that \(G^R (r, t, r') = [G^R (r, -t, r')]^*\).
The Fourier transforms
\[ G_R (rr'\varepsilon) = \int_0^\infty dt e^{i\varepsilon t} G_R (rt, r') \]
\[ G_A (rr'\varepsilon) = \int_{-\infty}^0 dt e^{i\varepsilon t} G_A (rt, r') \]
are analytic functions in the upper and lower halves of the complex planes. (To see this, substitute \( \varepsilon = \varepsilon' + i\varepsilon'' \) into the exponentials and check the convergence of the integrals.)

**Exercise 11** Show that on the real axis \((\varepsilon'' = 0)\), \(G_R (rr'\varepsilon) = [G_A (rr'\varepsilon)]^*\).

**Exercise 12** Show that the retarded and advanced Green’s functions of the free Fermi gas are given by
\[ G^{R,A} (\varepsilon, p) = \frac{1}{\varepsilon - \xi_p \pm i\delta}, \]
where \( \delta > 0 \).

Using the Sokhotsky formula
\[ \frac{1}{x - i\delta} = \mathcal{P} \frac{1}{x} + i\pi \delta (x), \]
where \( \mathcal{P} \) is the Cauchy principal part, we find that
\[ \text{Im} G^{R,A} = \mp \pi \delta (\varepsilon - \xi_p). \] (2.10)

This helps to establish a useful relation between the density of states and the Green’s function. For free fermions, the density of states (per one spin orientation) is defined as
\[ \nu (\varepsilon) = \int \frac{d^dp}{(2\pi)^d} \delta (\varepsilon - \xi_p). \]

Using (2.10), this formula can be re-written as
\[ \nu (\varepsilon) = \mp \frac{1}{\pi} \int \frac{d^dp}{(2\pi)^d} \text{Im} G^{R,A} (\varepsilon, \xi_p). \]

Even if fermions are not free, this relation still holds. This formula is very important in application to tunneling experiments. Quantity
\[ A (\varepsilon, \xi_p) = \mp \frac{1}{\pi} \text{Im} G^{R,A} (\varepsilon, \xi_p) \]
is called the *spectral function*. This is a very important characteristic of a many-body system. For free fermions, \( A (\varepsilon, \xi_p) \) is a \( \delta \)-function. This means that a fermion with momentum \( p \)
can be found in a state with one and only one energy \( \varepsilon = p^2/2m \). If fermions interact with each other or with phonons, etc., the \( \delta \)-function gets smeared. Quite often, it is replaced by a Lorentzian

\[
\delta (\varepsilon - \varepsilon_p) \rightarrow \frac{1}{\pi} \frac{\Gamma}{(\varepsilon - \varepsilon_p)^2 + \Gamma^2}.
\]

Now a fermion with momentum \( p \) can be found with any energy \( \varepsilon \). However, the probability of having a given energy \( \varepsilon \) is peaked at \( \varepsilon = p^2/2m \) and falls off outside the interval around this value. The width of this interval is determined by \( \Gamma \).

**Example 13** Another advantage of the retarded and advanced Green’s functions become obvious when one considers bi-linear combinations of these functions integrated over the momenta. Consider, for example

\[
A_{RR} = \int \frac{d^d p}{(2\pi)^d} G^R (\varepsilon_1, p) G^R (\varepsilon_2, p)
\]

\[
= \int_{0}^{\infty} d\varepsilon_p \nu (\varepsilon_p) G^R (\varepsilon_1, p) G^R (\varepsilon_2, p),
\]

where \( \nu (\varepsilon_p) \) is the density of states per one spin orientation. Switching to the new variable \( \xi_p = \varepsilon_p - \varepsilon_F \), we obtain for the integral

\[
A_{RR} = \int_{-\varepsilon_F}^{\varepsilon_F} d\xi_p \nu (\xi_p + \varepsilon_F) G^R (\varepsilon_1, p) G^R (\varepsilon_2, p).
\]

If \( |\varepsilon_1|, |\varepsilon_2| \ll \varepsilon_F \), we can extend the low limit of the integral to \(-\infty \) and take the density of states out of the integral

\[
A_{RR} = \nu (\varepsilon_F) \int_{-\infty}^{\infty} d\xi_p \frac{1}{(\varepsilon_1 - \xi_p + i\delta)(\varepsilon_2 - \xi_p + i\delta)} = 0.
\]

The integral equals to zero because both poles in variable \( \xi_p \) (\( \xi_p = \varepsilon_1 + i\delta, \xi_p = \varepsilon_2 + i\delta \)) are located in the same (upper) half-plane and one can always to draw the contour in the lower half-plane, where the integrand has no singularities. By the same token,

\[
A_{LL} = \int \frac{d^d p}{(2\pi)^d} G^L (\varepsilon_1, p) G^L (\varepsilon_2, p) = 0,
\]

whereas

\[
A_{RL} = A_{RL}^* = \frac{2\pi i \nu (\varepsilon_F) \frac{1}{\varepsilon_1 - \varepsilon_2 - i\delta}}{\varepsilon_1 - \varepsilon_2 + i\delta}.
\]

Notice that, since \( \delta \) is infinitesimally small, \( 2\delta \) and \( \delta \) can be considered as the same number.
Many-body systems are to be studied at finite temperatures. None of the Green’s functions introduced so far provides a convenient way to deal with finite temperatures. We need yet another Green’s function: the Matsubara one. The Matsubara Green’s function is defined similar to the time-ordered one, except for the time is imaginary

\[ G_M^{\alpha\beta}(\mathbf{r}, \mathbf{r}') = -\langle \langle \hat{T} \, \Psi_\alpha(\mathbf{r}) \, \bar{\Psi}_\beta(\mathbf{r}') \rangle \rangle. \]

\( \bar{\Psi}_\alpha(\mathbf{r}t) \) is the \( \hat{\Psi} \)-operator in the Matsubara representation

\[ \Psi_\alpha(\mathbf{r} \tau) = e^{\hat{H}' \tau} \bar{\Psi}_\alpha(\mathbf{r}) e^{-i\hat{H}' \tau}, \]

\[ \bar{\Psi}_\alpha(\mathbf{r} \tau) = e^{\hat{H}' \tau} \hat{\Psi}_\alpha^\dagger(\mathbf{r}) e^{-i\hat{H}' \tau}, \]

where \( \hat{H}' = \hat{H} - \mu \hat{N} \) is the ”Grand Canonical Hamiltonian” and \( \hat{T} \) is the ”tau-ordering operator”, defined similar to the \( \hat{T} \) operator in real time

\[ \hat{T}_\tau \hat{f}(\tau_1) \hat{g}(\tau_2) = \hat{f}(\tau_1) \hat{g}(\tau_2) \text{ for } \tau_1 > \tau_2 \]
\[ -\hat{g}(\tau_2) \hat{f}(\tau_1) \text{ for } \tau_1 < \tau_2 \]

\[ \hat{T}_\tau \hat{f}(\tau_1) \hat{g}(\tau_2) = \hat{f}(\tau_1) \hat{g}(\tau_2) \text{ for } \tau_1 > \tau_2 \]
\[ \hat{g}(\tau_2) \hat{f}(\tau_1) \text{ for } \tau_1 < \tau_2, \]

Notice that the Matsubara operator is obtained from the Heisenberg one by performing a Wick rotation: \( t = -i\tau \).

Finally, the double brackets \( \langle \langle . . \rangle \rangle \) mean taking the trace over quantum mechanical states and thermodynamic averaging with statistical weights. Explicitly,

\[ G_M^{\alpha\beta}(\mathbf{r}\tau, \mathbf{r}'\tau') = -\text{Tr} \left( \hat{T}_\tau e^{-\beta(\Xi-\hat{H}')} \Psi_\alpha(\mathbf{r}) \bar{\Psi}_\beta(\mathbf{r}') \right) = -e^{\beta \Xi} \text{Tr} \left( \hat{T}_\tau e^{\beta \hat{H}'} \bar{\Psi}_\alpha(\mathbf{r}) \Psi_\beta(\mathbf{r}') \right), \]

where \( \Xi \) is the Grand Canonical Potential (\( c- \) number) and \( \beta \equiv T^{-1} \).

**Claim 14** The most interesting property of the Matsubara Green’s function is that it is a periodic function of \( \tau \) in the interval \(-\beta \leq \tau \leq \beta \) for bosons and an anti-periodic function of \( \tau \) for fermions.

**Solution 15** Choose \( \tau < 0 \) and \( \tau' = 0 \). Then

\[ G^M(\mathbf{r}\tau, \mathbf{r}') = -\langle \langle \hat{T}_\tau \Psi(\mathbf{r}\tau) \bar{\Psi}(\mathbf{r}'0) \rangle \rangle = \mp e^{\beta \Xi} \text{Tr} \left( e^{-\beta \hat{H}'} \bar{\Psi}(\mathbf{r}') e^{\tau \hat{H}'} \Psi(\mathbf{r}) e^{-\hat{H}' \tau} \right). \]
As operators under the trace can be interchanged,

\[ G^M (r \tau, r') = \mp e^{\beta E} \text{Tr} \left( e^{\tau \hat{H}} \Psi (r) e^{-\beta \hat{H}} e^{-\beta \hat{H}} \bar{\Psi} (r') \right) \]

\[ = \mp e^{\beta E} \text{Tr} \left( e^{-\beta \hat{H}} e^{\tau \hat{H} + \beta} \Psi (r) e^{-\beta \hat{H} + \beta} e^{-\beta \hat{H}} \bar{\Psi} (r') \right) \]

\[ = \mp \langle \langle \Psi (r, \tau + \beta) \bar{\Psi} (r') \rangle \rangle = \pm G^M (r \tau + \beta, r') \]

Because of the (anti) periodicity, the Green’s function can be expanded into the Fourier series

\[ G^M (r \tau, r') = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} e^{-i \varepsilon_n \tau} G^M (r r', \varepsilon_n) = T \sum_{n=-\infty}^{\infty} e^{-i \varepsilon_n \tau} G^M (r r', \varepsilon_n). \]

Periodicity of \( G^M \) for bosons is ensured by choosing

\[ \varepsilon_n = 2\pi n/\beta = 2\pi n T, \quad n = 0, \pm 1, \pm 2 \ldots \quad (2.11) \]

For fermions,

\[ \varepsilon_n = \pi (2n + 1)/\beta = \pi (2n + 1) T, \quad n = 0, \pm 1, \pm 2 \ldots \quad (2.12) \]

\( G^M (r r', \varepsilon_n) \) is defined at discrete points on the imaginary axis of the complex plane.

**Exercise 16** Show that the Matsubara Green’s function for a free Fermi gas is given by

\[ G^M (\varepsilon_n, p) = \frac{1}{i \varepsilon_n - \xi_p}. \quad (2.13) \]

**Exercise 17** Show that the Matsubara Green’s function of the phonon field \( \phi \) is

\[ D^M (\omega_n, q) = -\frac{\omega_q^2}{\omega_n^2 + \omega_q^2}. \quad (2.14) \]

**Remark 18** Matsubara functions are useful not only at finite temperatures but also at \( T = 0 \). In the limit of \( T \to 0 \), Matsubara frequencies in Eqs. (2.11) and (2.12) become continuous variables, and the discrete Matsubara sums are replaced by the integrals

\[ T \sum_{\xi_n} \ldots \to \int \frac{d\xi}{2\pi} \ldots \]

**Example 19** Similar to Eqs.(2.6,2.7,2.8), Matsubara Green’s functions can be related to observable quantities. For example, the number density is obtained from the Matsubara Green’s as

\[ n = \pm G^M (r \tau, r) \big|_{\tau \to 0} \]

or

\[ n = \pm T \sum_{\xi_n} e^{-i \varepsilon_n \tau} \int \frac{dp}{(2\pi)^d} G^M (\varepsilon_n, p) \big|_{\tau \to 0}. \]
Example 20 Let’s calculate the following quantity for fermions

\[ P = T \sum_{\epsilon_n} G^M (\epsilon_n, p). \]

To get a guess at what this expression should be equal to, let’s first take the \( T \to 0 \) limit. Then,

\[
\begin{aligned}
P (T = 0) &= \int \frac{d\epsilon}{2\pi} \frac{1}{i\epsilon - \xi_p} = \int \frac{d\epsilon}{2\pi} \frac{-i\epsilon - \xi_p}{\epsilon^2 + \xi_p^2} \\
&= -i \int \frac{d\epsilon}{2\pi} \frac{\epsilon}{\epsilon^2 + \xi_p^2} - \xi_p \int \frac{d\epsilon}{2\pi} \frac{1}{\epsilon^2 + \xi_p^2}.
\end{aligned}
\]

The first integral diverges logarithmically but, as we can always choose the symmetric upper limits, it in fact equals to zero by parity. The second integral is elementary (just don’t forget that \( \xi_p \) may be of both signs, hence the result is not simply \( \pi/\xi_p \) but rather \( \pi/|\xi_p| \)), so that

\[ P (T = 0) = -\frac{1}{2} \text{sgn} \xi_p. \]

For \( T = 0 \), this is nothing more than \( n_p - 1/2 \). Indeed, for \( p < p_F \), \( \xi_p < 0 \) and \( n_p - 1/2 = 1 - 1/2 = 1/2 \); for \( p > p_F \), \( \xi_p > 0 \) and \( n_p - 1/2 = 0 - 1/2 = -1/2 \). The conjecture is then that

\[ n_p - \frac{1}{2} = T \sum_{\epsilon_n} G^M (\epsilon_n, p) \tag{2.15} \]

Exercise 21 Prove Eq.(2.15) by applying the Poisson summation formula.

F. Analytic continuation

Retarded and advanced Green’s function are obtained by analytic continuation of the Matsubara Green’s function from the discrete points on the imaginary axis to the entire complex planes. The details of the procedure are given in standard textbooks, see, e.g., Ref.[3]. The rule we need to know is

\[
\begin{align*}
G^R (\omega, p) &= G^M (\omega_n, p) |_{\omega_n - \omega + i\delta} \text{ for } \omega_n > 0 \\
G^A (\omega, p) &= G^M (\omega_n, p) |_{\omega_n - \omega - i\delta} \text{ for } \omega_n < 0.
\end{align*}
\]

Comparing Eqs.(2.9) and (2.13), we see that this rule is indeed obeyed.
III. FEYNMAN DIAGRAMS

I am going to skip all the details of the formal derivation of the diagrammatic technique (adiabatic switching of the interaction, interaction representation, $S-$matrix, etc.) and just formulate the diagrammatic rules. Suppose that we develop the perturbation theory for electron-phonon interaction. Graphically, the diagrams for the Green’s function are represented by the following sequence (see Fig.1). The solid line with an arrow corresponds the free Green’s function. e.g., Eq.(2.13), if we are using the Matsubara representation. The wavy line is the phonon Green’s function, Eq.(2.14), multiplied by the electron-phonon coupling constant $g$. The double solid line corresponds to the exact (renormalized) Green’s function. The direction of the arrow corresponds to the direction of time. The process depicted in Fig.1 corresponds to fixing the initial and final states as: initial–fermion $\varepsilon_n, \mathbf{p}$, no phonons; final–fermion $\varepsilon_n, \mathbf{p}$, no phonons. Electrons propagate through an elastic medium interacting with virtual phonons. This is why all phonon lines are always inserted into the electron lines. I grouped the diagrams by the order of the perturbation theory. The main rule of the diagram technique is the energy and momenta are conserved at every vertex, i.e., in every act of interaction.

For electron-electron (or, more generally, fermion-fermion) interaction, the diagrams are similar except for we have one additional element to first order: the tadpole diagram (diagram b) in Fig. 2). To second order, tadpoles proliferate and get mixed with other elements. The wavy line in tadpoles carries zero momentum, and this is why tadpoles are absent for the electron-phonon interaction: by construction, the electron-phonon vertex vanishes for $q \to 0$. In general, fermion-fermion interaction does not have to vanish in this limit. However, we can still get rid of tadpoles. The closed circle is a Green’s function at coinciding spatial and temporal arguments, i.e., the number density. Therefore, all tadpoles renormalized the fermion density at fixed chemical potential. If we are dealing with Coulomb interaction in a jellium model, the total charge density (electrons $+$ions) is equal to zero, so that tadpoles are eliminated. For He, the total density is not equal to zero but we can collect all tadpoles diagrams together and treat them separately. Once they are removed, the fermion-fermion diagrams are the same as electron-phonon ones, except for the wavy line is the interaction potential, $V(q)$.

Finally, the last important example of diagrammatic technique is the interaction of elec-
FIG. 1: Perturbative expansion of the Green’s function for the electron-phonon interaction.
FIG. 2: Perturbative expansion of the Green’s function for the fermion-fermion interaction.
FIG. 3: Perturbative expansion of the Green’s function in the electron-impurity interaction. Each circle corresponds to scattering by the exact (before averaging) impurity potential (3.1).

\[ \text{ FIG. 4: Perturbative expansion for the disorder-averaged Green’s function.} \]

electrons with static impurities. Imagine that we have \( N \) impurities scattered randomly over the sample. Impurity \( \#i \) is at point \( \mathbf{r}_i \). Each of the impurities is described by the potential \( U(\mathbf{r}) \). The total potential field on the electron is

\[ W(\mathbf{r}) = \sum_{i=1}^{N} U(\mathbf{r} - \mathbf{r}_i). \quad (3.1) \]

Corrections to the Green’s function for a fixed realization of impurities are shown in Fig.3. Now we can average the Green’s function over realizations of disorder, i.e., over the locations of impurities. To first order, we obtain the average impurity potential \( \langle W \rangle \). This can be always added to the reference energy. The non-trivial element, the correlation function of the impurity potential

\[ \langle W(\mathbf{r}) W(\mathbf{r}') \rangle = \langle \sum_{i,j} U(\mathbf{r} - \mathbf{r}_i) U(\mathbf{r}' - \mathbf{r}_j) \rangle \]

shows up to second order. Averaging means integrating over positions of all impurities. For \( i \neq j \) in the sum, the average is vanishingly small. Therefore, only diagonal elements \( i = j \) survive in the sum. Also, all impurities contribute equally to the average, so one can integrate over the position of the first impurity and multiply the result by \( N_i \)

\[ \langle W(\mathbf{r}) W(\mathbf{r}') \rangle = \langle \sum_{i} U(\mathbf{r} - \mathbf{r}_i) U(\mathbf{r}' - \mathbf{r}_i) \rangle \]

\[ = N_i \int \frac{d^d r_1}{\text{vol}} \int \frac{d^d q}{(2\pi)^d} \int \frac{d^d q'}{(2\pi)^d} U(\mathbf{q}) U(\mathbf{q}') e^{i(q(r-r_i)+q'(r'-r_i))} \]

\[ = n_i \int \frac{d^d q}{(2\pi)^d} \int \frac{d^d q'}{(2\pi)^d} \delta(\mathbf{q} + \mathbf{q}') U(\mathbf{q}) U(\mathbf{q}') e^{i(q r + q' r')} \]

\[ = n_i \int \frac{d^d q}{(2\pi)^d} U(\mathbf{q}) U(-\mathbf{q}) e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')} \]

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Transforming \( \langle W(r) W(r') \rangle \) into the Fourier space,

\[
\langle WW \rangle_q = n_i |U(q)|^2.
\]

This is the effective potential acting on average on an electron. \( \langle WW \rangle_q \) is denoted by a dashed line in Fig. 4. If the impurity potential is weak (Born approximation), we do neet to consider higher order correlators. Higher order diagrams are obtained by inserting a single dashed line into the Green’s function in all possible ways, as shown in Fig. 4.

### A. Self-energy

Consider all diagrams, e.g., for the electron-phonon interaction, which cannot be cut into two parts without cutting at least one interaction line. The first order diagram is of this type. All second-order diagrams except for diagram (b) also belong to this type. The sum of all such diagrams is depicted graphically by a circle, called the self-energy. The perturbation series can be re-drawn using a circle as shown in Fig. 5. By direct inspection one can see that all diagrams are reproduced in this way. Algebraically, the series for \( G \) can be written as

\[
G = G_0 + G^2_0 \Sigma + G^3_0 \Sigma^2 \cdots
\]

This is equivalent to the Dyson equation

\[
G = G_0 + G_0 \Sigma G.
\]

Solving this equation, we obtain

\[
G = \frac{1}{G_0^{-1} - \Sigma}.
\]

For example, in Matsubara representation

\[
G^M = \frac{1}{i \varepsilon - \xi_p - \Sigma^M (\varepsilon, \xi_p)}.
\]
Similarly, retarded (advanced) Green’s function is
\[ G^{R,A} = \frac{1}{\varepsilon - \xi_p - \Sigma^{R,A}(\varepsilon, \xi_p)}. \] (3.2)

Certainly, a solution of the Dyson equation is not very instructive as long as we have not found \( \Sigma \), which can be done only in the approximate way. Nevertheless, this parameterization allows to introduce such important concepts as the effective mass and damping of single-particle states. We can also immediately specify the general conditions for the system to be a Fermi liquid.

B. Effective mass and decay rate

The main premise of the Fermi-liquid theory is that the system of interacting fermions behaves similarly to a system of weakly interacting quasi-particles. At the lowest energy scales (as counted from the Fermi energy) these quasiparticles become free, and then the similarity becomes complete. The masses of these quasiparticles are, generally speaking, different from the bare masses. Interacting fermions will behave as effectively free particles only if their Green’s function have the same form as free-fermion Green’s function. This statement is equivalent to the requirement that the self-energy entering, e.g. Eq.(3.2) allows for a expansion in both of its arguments, the linear terms being the most important ones
\[ \Sigma^R(\varepsilon, \xi_p) = \Sigma_0 - \lambda_\varepsilon \varepsilon + \lambda_p \xi_p + O(\varepsilon^2, \xi_p^2) + \ldots \] (3.3)
The constant term (\( \Sigma_0 \)) can be absorbed in a shift of the chemical potential. Doing so and keeping only the leading terms, the Green’s function can be written as
\[ G^R(\varepsilon, \xi_p) = \frac{1 + \lambda_\varepsilon}{\varepsilon - \xi_p + \lambda_\varepsilon + i\delta} \equiv \frac{Z}{\varepsilon - \xi_p^* + i\delta} \]
where \( Z \) is the called the renormalization factor or single-particle residue. \( Z = 1 \) corresponds to the Fermi gas, finite \( Z \) in the interval \( 0 < Z < 1 \) to the Fermi liquid.

Remark 22 Finite \( Z \) means that the occupation number has a discontinuity at the Fermi surface. Indeed, switching to Matsubara representation, the occupation number of an interacting Fermi system at \( T = 0 \) is given by
\[
n_p - \frac{1}{2} = \int \frac{d\varepsilon}{2\pi} \frac{Z}{i\varepsilon - \xi_p^*} = -\frac{1}{2} Z \text{sgn}(\xi_p^*). \]
FIG. 6: Self-energy diagrams for the electron-phonon and fermion-fermion interaction (after the tadpoles diagrams were removed).
The discontinuity in $n_p$ is
\[ n_{p=p_F-\delta} - n_{p=p_F+\delta} = Z. \]

Vanishing $Z$ means smearing of the Fermi surface.

Recalling that near the Fermi surface $\xi_p = (p^2 - p_F^2)/2m \approx (p - p_F) v_F$, we see that $\xi_p^*$ contains a renormalized Fermi velocity
\[ v_F^* = v_F \frac{1 + \lambda_p}{1 + \lambda_\varepsilon}. \]  

In the Fermi-liquid theory, the effective mass is defined as
\[ m^* = \frac{p_F}{v_F} = m \frac{1 + \lambda_\varepsilon}{1 + \lambda_p}, \]  

where $m$ is the bare mass. The Fermi momentum is the same in the interacting and free system. More generally, the volume of the Fermi surface is independent of the interaction. This very non-trivial statement, known as the Luttinger theorem, can be understood from the Pauli principle: for fixed number of particles, the volume of the Fermi surface is determined only by this number.

Remark 23 Notice that this effective mass has nothing to do with band mass defined as $m_b = (d^2 \varepsilon_p/dp^2)^{-1} |_{p=p_F}$. Renormalization of the effective mass in the Fermi liquid is, in fact, renormalization of the Fermi velocity, i.e., of the slope $d\varepsilon_p/dp$.

According to Eq.(3.5), the effective mass is finite as long as
\[-1 < \lambda_\varepsilon < \infty \] \[-1 < \lambda_p < \infty. \]

Cases of $\lambda_\varepsilon = \infty$ and $\lambda_p = -1$ correspond to infinite effective mass, cases of $\lambda_p = \infty$ and $\lambda_\varepsilon = -1$ to zero effective mass. Finite effective mass means, among other things, that the specific heat remains linear in $T$, as it is for a Fermi gas. This can be understood as the renormalization of the density of states
\[ C(T)_{\text{Fermi gas}} = \frac{2\pi^2}{3} \nu(\varepsilon_F) T = \frac{1}{3} m p_F T \]
\[ C(T)_{\text{Fermi liquid}} = \frac{1}{3} m^* p_F T. \]
**Example 24** In non-Fermi liquids, expansion of $\Sigma$ starts with sublinear, non-analytic terms. For example,

$$\Sigma(\varepsilon) = -\alpha \varepsilon^a, a < 1.$$  

Formally, this means that coefficient $\lambda_\varepsilon$ is a not a constant any more but a function of $\varepsilon$: $\lambda_\varepsilon = \alpha \varepsilon^{a-1}$. The effective mass can be still defined as

$$m^* = (1 + \lambda_\varepsilon) m = \left(1 + \alpha \varepsilon^{a-1}\right) m$$

but now it is a constant but a function of $\varepsilon$ as well. Furthermore, $m^*$ diverges for $\varepsilon \to 0$. This means that the specific heat coefficient $C(T)/T$ is proportional to the $T$ dependent mass and diverges at $T \to 0$ as well

$$C(T)/T \propto T^{-(1-a)}.$$  

The divergence of the specific heat coefficient is hallmark of a non-Fermi-liquid behavior.

**Next-order terms in expansion (3.3) are imaginary.**

**Example 25** For the case of fermion-fermion interaction in 3D,

$$\Sigma^R(\varepsilon, \xi_p) = -\lambda_\varepsilon \varepsilon + \lambda_p \xi_p - i(1 + \lambda_\varepsilon) \gamma \varepsilon^2 + \ldots \quad (3.6)$$

The spectral function is now a Lorentzian

$$A(\varepsilon, p) = -\frac{1}{\pi} \text{Im} G^R(\varepsilon) = \frac{1}{\pi} \frac{Z \Gamma(\varepsilon)}{(\varepsilon - \xi_p)^2 + \Gamma^2(\varepsilon)}.$$  

For fixed $\varepsilon$, the width of the Lorentzian $\Gamma(\varepsilon) = \gamma \varepsilon^2$ varies quadratically with $\varepsilon$. For small energies, $\Gamma(\varepsilon) \ll \varepsilon$, the spectral function has a sharp peak. This is a typical behavior for the Fermi liquid.

**Definition 26** Condition $\Gamma(\varepsilon) \ll \varepsilon$ is usually taken as an operational definition of the Fermi liquid. Notice that the quadratic behavior of $\Gamma$ is a sufficient but not necessary condition for the Fermi liquid. The quasiparticles are well-defined as long as $\Gamma \propto \varepsilon^b$ with $b > 1$.

**Remark 27** The self-energy is an analytic function of $\varepsilon$. As for any analytic function, its real and imaginary parts are related to each other via a Kramers-Kronig relations

$$\text{Re} \Sigma^R(\varepsilon) = \frac{1}{\pi} \int d\varepsilon' \frac{\text{Im} \Sigma^R(\varepsilon')}{\varepsilon' - \varepsilon} \quad (3.7)$$

$$\text{Im} \Sigma^R(\varepsilon) = -\frac{1}{\pi} \int d\varepsilon' \frac{\text{Re} \Sigma^R(\varepsilon)}{\varepsilon' - \varepsilon}. \quad (3.8)$$
Because of these relations, the behaviors of real and imaginary parts of $\Sigma$ are correlated. Here are some simple rules relevant for the non-Fermi liquid behavior

\[
\text{Re} \Sigma^R(\varepsilon) \propto |\varepsilon| \ln |\varepsilon| \iff \text{Im} \Sigma^R(\varepsilon) \propto |\varepsilon| \\
\text{Re} \Sigma^R(\varepsilon) \propto \text{sgn} \varepsilon |\varepsilon|^a \iff \text{Im} \Sigma^R(\varepsilon) \propto |\varepsilon|^a \quad \text{for } 0 < a < 1.
\]

The first case is known as “marginal Fermi liquid”. In this case, quasiparticles are marginally defined ($\Gamma \sim \varepsilon$) but the effective mass diverges logarithmically. The second case tells us that the exponent in the power-law divergence of the effective mass gives us the exponent of the decay rate.

**Remark 28** Notice that $\text{Re} \Sigma^R(\varepsilon)$ is an odd function of $\varepsilon$ whereas $\text{Im} \Sigma^R(\varepsilon)$ is an even function of $\varepsilon$. This is easy to understand. Indeed, $\text{Im} \Sigma^R(\varepsilon)$ gives the decay rate of single-particle states. The rate must be the same for $\varepsilon > 0$ and $\varepsilon < 0$, therefore $\text{Im} \Sigma^R(\varepsilon)$ must be even in $\varepsilon$. Now, knowing that $\text{Im} \Sigma^R(\varepsilon)$ is even, we obtain from Eq. (3.7)

\[
\text{Re} \Sigma^R(-\varepsilon) = \frac{1}{\pi} \int d\varepsilon' \frac{\text{Im} \Sigma^R(\varepsilon')}{\varepsilon' + \varepsilon} = \frac{1}{\pi} \int d\varepsilon' \frac{\text{Im} \Sigma^R(\varepsilon')}{-\varepsilon' + \varepsilon} = -\text{Re} \Sigma^R(-\varepsilon).
\]


[2] The last term is often referred to as "four-fermion interaction”. (In fact, this is a two-fermion interaction but there are two fermions in the initial state and two in the final, hence total number of the $\Psi$-operators is four.