Chapter 22: Electric Field
Electric Field of Single Point Charge

The electric field from an isolated positive charge:

\[ \vec{E} = \frac{kq}{r^2} \hat{r} \]

The electric field from an isolated negative charge:

\[ \vec{E} = -\frac{kq}{r^2} \hat{r} \]
Example: Electric Field on Proton

➔ At surface of proton
  ◆ \( q = e = 1.6 \times 10^{-19} \text{ C} \)
  ◆ \( r = 10^{-15} \text{ m} \)

\[
E = \frac{kq}{r^2} = \frac{(9 \times 10^9)(1.6 \times 10^{-19})}{(10^{-15})^2} = 1.44 \times 10^{21} \text{ N/C}
\]

➔ \( E \) points radially outward for + charge
E Field of Two Equal, Positive Point Charges
E Field of Two Equal, Unlike Point Charges
Field Between Two Charged Parallel Plates

- Assume plates are much larger than separation
  - $E$ is approx. constant between plates
  - $E$ is zero outside the plates
  - This is a capacitor!

- $E$ points from + plate to – plate

- We will calculate $E$ in Chap. 23
  - Gauss’ law
  - Proportional to surface charge density
1. Rank magnitude of $E$ at $P_1$, $P_2$, $P_3$. Assume charges on rings are $+q$ and $+q$. 
Answer to Question #1

- $P_1$ has $E = 0$ since it is equidistant from ring A and B and they are same sign.
- $P_3$ has largest $E$ because it has contributions from ring A and B.
- $P_2$ has no contribution from ring B because it is at the center, thus it is only affected by ring A.
- So the order (smallest $E$ to largest $E$) is $P_1$, $P_2$, $P_3$. 
2. Rank magnitude of $E$ at $P_1$, $P_2$, $P_3$. Assume charges on rings are $+q$ and $-q$. 
Answer to Question #2

- $P_1$ has largest $E$ field since it is equidistant from ring A and B and their $E$ contributions add, rather than cancel, as in the first question.

- Hard to rank $E$ field of $P_3$ and $P_2$, in my opinion. Relative distances from the two rings are different and there is a cancellation in $P_3$. 
Calculate E of Dipole (⊥ axis)

➔ At point $x$, $E_x = 0$ and $E_y < 0$. Why?

\[
E_y = -\frac{2kQ}{r^2} \sin \theta = 2\left(\frac{-kQ}{x^2 + d^2/4}\right) \frac{d/2}{\sqrt{x^2 + d^2/4}} = \frac{-kQd}{\left(x^2 + d^2/4\right)^{3/2}}
\]

\[
E_y \approx -\frac{kp}{x^3} \quad x \gg d \quad p = Qd \quad \text{(dipole moment)}
\]
Calculate E of Dipole (along axis)

At point \( x \), \( E_x > 0 \) and \( E_y = 0 \). Why?

\[
E_x = \frac{kQ}{(x - d/2)^2} - \frac{kQ}{(x + d/2)^2} = \frac{2kQxd}{(x^2 - d^2/4)^2}
\]

\[
E_x \approx \frac{2kp}{x^3} \quad x \gg d \quad p = Qd \text{ (dipole moment)}
\]
Finding E Field from Charge Distribution

Perform integral over charge distribution

- Each component must be calculated separately (vector addition)

\[ dE_y = \frac{k dq}{r^2} (\sin \theta \text{ or } \cos \theta) \]

General helpful rules

- Use symmetry to see if any component must be zero
- Use symmetry to see if any component is doubled, etc.
- Express dq, r and trig functions in terms of “natural” variables defined by the problem
- Then we can integrate!
Center of Uniformly Charged Circle

- $E$ field is down. Why?
  - Uniform distribution of charge
  - Express $dq$, $r$, $\sin \theta$ in terms of $\theta$
  - Top, bottom give same contribution

$\theta$

$dq = \lambda \, ds = \lambda \, rd\theta$

$E_y = 2 \times \int_0^\pi - \frac{k \lambda \, rd\theta}{r^2} \sin \theta$

$= \frac{-2k \lambda}{r} \left( -\cos \theta \right)_0^\pi$

$= \frac{-4k \lambda}{r}$

$\lambda = \frac{q}{\pi r}$

$E_y = -\frac{4kq}{\pi r^2}$
Axis of Uniform Charged Ring (+Q)

- Point is distance $z$ above center of charged ring, radius $R$
  
  - Uniformly charged (ind. of angle $\phi$)
  - Horizontal components cancel

\[
dE_z = \frac{k dq \sin \theta}{r^2} = \frac{k(\lambda R dq)}{z^2 + R^2} \sin \theta
\]

\[
\sin \theta = \frac{z}{r} = \frac{z}{\sqrt{z^2 + R^2}}
\]

\[
E_z = \int_0^{2\pi} \frac{k \lambda z R dq}{(z^2 + R^2)^{3/2}} = \frac{kz 2\pi \lambda R}{(z^2 + R^2)^{3/2}}
\]

\[
E_z = \frac{k Q z}{(z^2 + R^2)^{3/2}}
\]

Check this!
Behavior at Large and Small $z$

- **Exact**
  \[ E_z = \frac{kQz}{(z^2 + R^2)^{3/2}} \]

- **For $z$ small**
  \[ E_z \to \frac{kQz}{R^3} \]
  Becomes 0 when $z = 0$

- **For $z$ large**
  \[ E_z \to \frac{kQ}{z^2} \]
  Coulomb’s law
  As expected
Uniformly Charged Line

- Distance $y$ above midpoint of charged line of length $L$
  - Uniformly charged
  - $E_x$ components cancel

\[ \lambda = \frac{Q}{L} \]

\[ dq = \lambda \, dx \]

\[ E_y = \int \frac{kdq}{r^2} \sin \theta = \int_{-L/2}^{L/2} \frac{k(\lambda \, dx)}{y^2 + x^2} \frac{y}{\sqrt{y^2 + x^2}} = \int_{-L/2}^{L/2} \frac{ky(\lambda \, dx)}{(y^2 + x^2)^{3/2}} \]

\[ E_y = \frac{k \lambda L}{y \sqrt{y^2 + L^2 / 4}} \]

Check this!
Special Cases for Charged Line

- **Exact expression**
  \[ E_y = \frac{k\lambda L}{y\sqrt{y^2 + L^2 / 4}} \]

- **Infinite line (L → ∞)**
  \[ E_y = \frac{2k\lambda}{y} \]
  Falls as inverse distance

- **Zero length (point)**
  \[ E_y = \frac{k\lambda L}{y^2} = \frac{kQ}{y^2} \]
  Coulomb’s law
  As expected
Uniformly Charged Disk

Point is distance $z$ above axis of charged disk, radius $R$

\[ E_z = \int \frac{k dq}{r^2} \sin \theta \]

\[ = \int_0^R \int_0^{2\pi} \frac{k (\sigma \rho d \rho d\phi)}{z^2 + \rho^2} \frac{z}{\sqrt{z^2 + \rho^2}} \]

\[ = \int_0^R k \left(2\pi z \sigma d\rho \right) \frac{1}{(z^2 + \rho^2)^{3/2}} \]

\[ E_z = 2\pi k \sigma \left(1 - \frac{z}{\sqrt{z^2 + R^2}} \right) \]
Charged Disk (cont)

⇒ Exact expression

\[ E_z = 2\pi k \sigma \left( 1 - \frac{z}{\sqrt{z^2 + R^2}} \right) \]

⇒ \( z \ll R \)

\[ E_z \rightarrow 2\pi k \sigma = \frac{\sigma}{2\varepsilon_0} \]

Independent of \( z \)!
See next chapter

⇒ \( z \gg R \)

\[ E_z \rightarrow \frac{k\pi R^2}{z^2} \sigma = \frac{kQ}{z^2} \]

Coulomb’s law
As expected