Exam 1 Solutions

1. A central particle of charge $-3q$ is surrounded by two circular rings (radius $R$ and $2R$, respectively) of charged particles, as shown in the figure. What are the magnitude and direction of the net electrostatic force on the central particle due to the other particles?

Most charges are distributed symmetrically around the circle, so that the net force on the central particle is zero. The only unbalanced forces lie along the $y$ axis from 3 charges:

$$F_y = -k \frac{(-3q)(+2q)}{R^2} - k \frac{(-3q)(+4q)}{4R^2} + k \frac{(-3q)(+6q)}{4R^2}$$

$$F_y = k \frac{6q^2}{R^2} + k \frac{3q^2}{2R^2} - k \frac{9q^2}{2R^2} = k \frac{9q^2}{2R^2} \text{ up}$$

2. Charge is uniformly distributed around a ring of radius 10cm and the electric field $E$ is measured along the ring's central axis (perpendicular to the plane of the ring. At what distance (in cm) from the ring's center is the magnitude of $E$ maximized?

The electric field from a continuous charge distribution is solved by integration. Keep in mind that by symmetry, only the field pointing along the axis perpendicular to the ring plane, let's call it the $z$ axis, is non-zero.

$$dE_z = \frac{dq}{\sqrt{R^2 + z^2}} \cos \theta \quad \cos \theta = \frac{z}{\sqrt{R^2 + z^2}}$$

$$dE_z = \frac{\lambda ds z}{(R^2 + z^2)^{3/2}}$$

$$\Rightarrow E_z = \frac{2\pi R \lambda z}{(R^2 + z^2)^{3/2}}$$

To find where the field is a maximum, take a derivative:
\[
\frac{dE_z}{dz} = 0 = \left(R^2 + z^2\right)^{-3/2} + z\left(-\frac{3}{2}\right)\left(R^2 + z^2\right)^{-5/2} \ (2z)
\]
\[
0 = R^2 + z^2 - 3z^2
\]
\[
z = \frac{R}{\sqrt{2}} = 7.1 \text{ cm}
\]

3. Three concentric conducting shells have radii \(R_1 < R_2 < R_3\) and charges \(+2Q\), \(-4Q\) and \(+5Q\), respectively. What is the charge on the inner surface of the outermost shell?

*The electric field must be zero inside an isolated conductor. So a spherical Gaussian surface within each conductive shell must have zero flux since \(E=0\), which implies that by Gauss’ Law the enclosed charge must be zero. Thus, the inside surface of each shell must have an electric charge that exactly balances the charge from the inside conductors. Therefore, the inside surface of the outermost shell must balance a charge of \(-2Q\), which is thus \(+2Q\).*

4. Two charged metal spheres are connected by a copper wire, as shown in the figure. Note that \(r_A = 2r_B\) and the spheres are far apart. What is the ratio \(\sigma_A / \sigma_B\) of surface charge densities?

*The electric potential must be a constant for a conductive surface. Thus, the two surfaces connected by a copper wire must be at the same electric potential (if they were not, charge would flow). Each sphere has a different charge, but the electric potential from each charged sphere (the same as for a point charge) must be the same:*

\[
V = k \frac{Q_A}{r_A} = k \frac{Q_B}{r_B}
\]
\[
r_B = \frac{r_A}{2} \Rightarrow k \frac{Q_A}{r_A} = 2k \frac{Q_B}{r_A}
\]
\[
\Rightarrow Q_B = Q_A / 2
\]

*Surface charge densities are defined by:*
\[ \sigma = \frac{Q}{A} \text{ where } A = \text{area} \]

\[ \Rightarrow \sigma_A = \frac{Q_A}{4\pi r_A^2} \]

\[ \sigma_B = \frac{Q_B}{4\pi r_B^2} = \frac{Q_A/2}{4\pi r_A^2/4} = 2 \frac{Q_A}{4\pi r_A^2} \]

\[ \Rightarrow \frac{\sigma_A}{\sigma_B} = \frac{1}{2} \]

5. Two parallel plate capacitors, \( C_1 = 20 \mu F \) and \( C_2 = 10 \mu F \), respectively, are connected in series to a battery. Which statement is true?

*For capacitors in series, the magnitude of the charge on each plate must be the same: \( Q_1 = Q_2 \). The reason is that the conductor between the two capacitors must have zero net charge.*

6. Particles 1, with a charge \( q_1 \), and 2, with a charge \( q_2 \), are on the x-axis with particle 1 at \( x = a \) and particle 2 at \( x = -2a \). For the net force on a third charged particle placed at the origin to be zero, \( q_1 \) and \( q_2 \) must be related by \( q_2 = : \)

*Set the net force on the third particle to be zero along the x-axis:*

\[ F_{31} + F_{32} = 0 = k \frac{q_1 q_3}{a^2} - k \frac{q_2 q_3}{4a^2} \]

\[ \Rightarrow q_2 = 4q_1 \]

7. Consider a cube with side length \( a = 5 \text{ m} \), as shown in the diagram. Suppose that an electric field is present and points in the negative z direction (downward) and has the following form: \( \vec{E} = -bz \hat{z} \), where \( b = 4.5 \times 10^3 \text{ N/(C m)} \). What is the total charge contained within the cube (in \( \mu C \))?

*The electric field points along the z-axis, so the electric flux, defined by \( \Phi = \vec{E} \cdot \vec{A} \), is non-zero only for the top and bottom faces. But the bottom face at \( z = 0 \) has \( E = 0 \). Thus, all the flux comes from the top face. The electric field points into the cube (downward), which is opposite to the area vector for that top face so that the flux is negative. By Gauss’ Law, one can solve for the charge enclosed:*
8. Consider a circular plastic rod with charge \( +q \) uniformly distributed along the top left quadrant and charge \( -q \) uniformly distributed along the bottom left quadrant as shown. What is the direction of the electric field at point P, the center of the semicircle?

The electric field points away from positive charge, and toward negative charge. By symmetry, the electric field from the \( +q \) arc points in the SE direction (if \( +y \) points in the North direction on a compass). The electric field from the \( -q \) arc points in the SW direction with exactly the same magnitude. Thus, the \( x \) components of the electric fields cancel in the sum, and only the field in the negative \( y \) direction (down) survives.

9. A parallel plate capacitor with a capacitance of 3\( \text{F} \) is connected to a 5\( \text{V} \) battery. If the plates are squeezed to one-third of the original separation, how much work must be done by the battery to maintain the same potential difference across the plates?

The stored potential energy of a capacitor is \( U = \frac{1}{2} CV^2 \). The capacitance of a parallel-plate capacitor satisfies: \( C = \varepsilon_0 \frac{A}{d} \), so if the plates get 3 times closer, the capacitance is 3 times larger \( \Rightarrow 9\text{F} \). Thus the potential energy increases since the voltage remains the same across the capacitor but the capacitance increases. The net work done by the battery to change this potential energy is:

\[
W = \Delta U = U_f - U_i = \frac{1}{2} V^2 \left( C_f - C_i \right)
\]

\[
\Delta U = \frac{1}{2} (5 \text{ V})^2 (9 - 3 \text{ F}) = 75 \text{ J}
\]

10. The electric potential varies linearly along the \( x \)-axis from 24\( \text{V} \) at \( x = 3 \text{ m} \) to 12\( \text{V} \) at \( x = 7 \text{ m} \). What is the magnitude of the force acting on an electron at \( x = 3 \text{ m} \)?

\[
E = -\frac{\Delta V}{\Delta s} = -\frac{12 - 24 \text{ V}}{7 - 3 \text{ m}} = 3 \text{ V/m}
\]

\[
F = qE = \left(-1.6 \times 10^{-19} \text{ C}\right)(3 \text{ V/m})
\]

so \( |F| = 4.8 \times 10^{-19} \text{ N} \)