Exam 2 Solutions

Note that there are several variations of some problems, indicated by choices in parentheses.

**Problem 1**

Part of a long, straight insulated wire carrying current $i$ is bent into a circular section of radius $d$ as shown in the figure. What is the magnitude of the magnetic field at the center of circular section if both straight and curved sections lie in the plane of the page as shown?

![Diagram of a wire and a circular section](image)

1. $(\mu_0 i / 2 \pi d) (\pi - 1)$
2. $(\mu_0 i / 2 \pi d) (\pi + 1)$
3. $(\mu_0 i / 2 \pi d)$
4. $(\mu_0 i / 2 d)$
5. 0

The field contribution from the straight wire at the center of the circular points into the page with magnitude $\frac{\mu_0 i}{2\pi d}$.

The field contribution from the loop at its center points out of the page. You can use the Biot-Savart Law to compute it:

$$dB = \frac{\mu_0 i}{4\pi} \frac{ds \times \hat{r}}{r^2} = \frac{\mu_0 i}{4\pi} \frac{r d\phi}{r^2}$$

$$\Rightarrow B = \frac{\mu_0 i}{2d}$$

So the magnitude is the difference:

$$B_{tot} = \frac{\mu_0 i}{2\pi d} (\pi - 1)$$
Problem 2

A circular loop of radius 0.1 m is oriented in the xy-plane and carries a current \( i = 2 \text{A} \) in the counter-clockwise direction as shown. It is allowed to rotate about the x-axis. A uniform magnetic field \( \mathbf{B} \) of magnitude 0.3 T is applied in the y direction. What is the magnitude of the torque (in N-m) about the x-axis, and in what direction does the top part of the shown loop move?

\( \mathbf{B} \)

(1) 0.02, into page  
(2) 0.02, out of page  
(3) 0.06, into page  
(4) 0.06, out of page  
(5) 0, does not move

\[ \mathbf{\mu} = i \mathbf{A} \quad \text{points out page} \]
\[ \mathbf{\tau} = \mathbf{\mu} \times \mathbf{B} \quad \text{points in } -\hat{i} \text{ direction} \]
\[ \mathbf{\tau} = i \pi r^2 B = 0.02 \quad \text{N-m} \]

Since the torque is in the \(-i\) direction, the top of the loop moves into the page (also can be worked out from \( d\mathbf{F} = id\mathbf{A} \times \mathbf{B} \)).

Problem 3

An electron with a velocity vector of \( \mathbf{v} = 10,000 (i+j) \text{m/s} \) moves through a uniform magnetic field given by \( \mathbf{B} = 4 (-i + j) \text{T} \). What is the force (in N) on the electron?

(1) \(-1.3 \times 10^{-14} \quad \mathbf{k}\)  
(2) \(1.3 \times 10^{-14} \quad \mathbf{k}\)  
(3) \(-6.4 \times 10^{-15} \quad \mathbf{k}\)  
(4) \(6.4 \times 10^{-15} \quad \mathbf{k}\)  
(5) \(-40,000 \quad \mathbf{j}\)

\[ \mathbf{F} = q \mathbf{v} \times \mathbf{B} \]
\[ = -4(10000)e(\hat{i} + \hat{j}) \times (-\hat{i} + \hat{j}) \]
\[ = -40000e(\hat{i} \times \hat{j} - \hat{j} \times \hat{i}) \]
\[ = -80000e\mathbf{k} \]
\[ = -1.3 \times 10^{-14} \mathbf{k} \]
Problem 4

A power line that is horizontal to the surface of the Earth carries a current of 9000 A from north to south. Earth’s magnetic field of magnitude 60 μT is directed toward the north and is inclined downward from horizontal at an angle of 60°. What is the magnitude of the force per unit length on the power line and its direction?

(1) 0.47 N/m, east
(2) 0.47 N/m, west
(3) 0.54 N/m, west
(4) 0.54 N/m, east
(5) 0.27 N/m, south

The force on a wire carrying a current in a magnetic field is given by:
\[ \mathbf{F} = i \mathbf{l} \times \mathbf{B} \]
For the drawing shown, this is going to lead to a force coming out of the page. This would be in the eastern direction. The magnitude of the force per unit length is:
\[ \frac{F}{l} = (9000)(60 \times 10^{-6}) \sin 120° = 0.47 \text{N} \]

Problem 5

In the circuit shown, \( \varepsilon_1 = 3.0 \text{V} \) and \( \varepsilon_2 = \varepsilon_3 = 1.5 \text{V} \). Also \( R_1 = 2 \Omega \) and \( R_2 = R_3 = 1 \Omega \). What is the magnitude of the current flowing through \( R_2 \)?

(1) 0.3 A (2) 1.5 A (3) 0.6 A (4) 0.9 A (5) 0
Let each current point vertically up through each EMF source. Since $\varepsilon_2 = \varepsilon_3$ and $R_2 = R_3$, the current $i_2 = i_3$ (the branches are equivalent).

By the junction rule we have $i_1 = -2i_2$
Then by the loop rule for the outermost loop:

$$\varepsilon_1 - R_1i_1 + R_2i_2 - \varepsilon_2 = 0$$
$$\left(\varepsilon_1 - \varepsilon_2\right) + 2R_1i_2 + R_2i_2 = 0 \quad \text{plug in junction rule}$$
$$\left(\varepsilon_1 - \varepsilon_2\right) = -2R_1i_2 - R_2i_2$$
$$i_2 = \frac{\left(\varepsilon_1 - \varepsilon_2\right)}{2R_1 + R_2} = \frac{1.5}{5} = -0.3$$

**Problem 6**

What is the equivalent resistance of the shown circuit as seen by the EMF source if all resistors have resistance $R$?

![Circuit Diagram]

(1) $3R/5$  (2) $R/4$  (3) $R$  (4) $4R$  (5) $5R/2$

The 2 resistors on the right are in parallel with equivalent resistance $R/2$. These are in series with $R$ at the top, for resistance $3R/2$. That in turn is in parallel with a resistance $R$, so the total equivalent resistance is $3R/5$.

**Problem 7**

The current density $\mathbf{j}$ inside a long, solid, cylindrical wire of radius $a$ is in the direction of the central axis, and its magnitude varies inversely with radial distance $r$ from the axis according to $\mathbf{j} = j_0 a/r$. Find the magnitude of the magnetic field at $r = a/2$.

(1) $\mu_0 j_0 a$  (2) $\mu_0 j_0 a/2$  (3) $\mu_0 j_0 a/3$  (4) $\mu_0 j_0 a/4$  (5) $\mu_0 j_0 a \ln(a/2)$
Use Ampere’s law. By symmetry, the magnetic field is directed axially around the wire, and its magnitude depends only on the distance from the central axis. We can best exploit this by choosing the closed path to be a circle of radius \( r \), perpendicular to the central axis and centered on it. The boundary of such a circle is its circumference so \( \int ds \cdot \mathbf{B} = 2\pi r B(r) \)

By Ampere’s law this must be \( \mu_0 2\pi \int_0^r dr' r' j(r') \)

so \( B(r) = \frac{\mu_0}{r} \int_0^r dr' r' j(r) \)

Now use the fact that \( j(r) = j_0 \frac{a}{r} \) and integrate.

For \( r < R \) we have \( B(r) = \frac{\mu_0}{r} j_0 a r = \mu_0 j_0 a \) independent of radius \( r \).

**Problem 8**

To create a magnetic field of 2T inside of a solenoid of length 50cm and radius 1cm, how many total windings are required for the entire length if the current in the wire cannot exceed 10A?

(1) 80,000 (2) 160,000 (3) 1,600 (4) 10,000 (5) 800,000

The field inside of a solenoid can be found by Ampere’s Law and is:

\[
B = \mu_0 n i = \mu_0 \frac{N}{l}
\]

\[
\Rightarrow N = \frac{B l}{\mu_0 i} = \frac{2(0.5)}{10(4\pi \times 10^{-7})} = 80,000
\]

**Problem 9**

A proton of velocity \( 6 \times 10^7 \) m/s and mass \( 1.67 \times 10^{-27} \) kg is to be kept in a circular orbit using a magnet. What is the minimum diameter of the magnet necessary assuming that it provides a uniform magnetic field of 0.5T between its pole tips.

(1) 2.5m (2) 1.2m (3) 6.5\times10^7m (4) 1.6m (5) 0.8m

The relation between the momentum and the radius of curvature of a charged particle in a uniform magnetic field is given by:

\[
mv = qBr
\]

\[
\Rightarrow d = 2r = \frac{2mv}{qB} = \frac{2(1.67 \times 10^{-27})(6 \times 10^7)}{(1.6 \times 10^{-19})(0.5)}
\]

\[
d = 2.5m
\]
Problem 10
An initially uncharged capacitor and a 100 kΩ resistor are connected in series with an EMF source at t=0 to form a complete circuit. If the capacitor is charged to half of its maximum value in 0.1s, what must be the capacitance of the capacitor?

(1) 1.4 µF  (2) 1.0 µF  (3) 6.9×10^5 F  (4) 0.5 µF  (5) 0.5 F

The charge on the capacitor in an RC circuit increases in time as:

\[ q(t) = C \varepsilon (1 - e^{-t/\tau_{RC}}) \]

where \( \tau_{RC} = RC \)

This reaches half of its maximum when

\[ q(t) = \frac{1}{2} C \varepsilon = C \varepsilon (1 - e^{-t/\tau_{RC}}) \]

\[ \Rightarrow e^{-t/\tau_{RC}} = \frac{1}{2} \]

\[ \frac{t}{RC} = \ln 2 \]

\[ C = \frac{t}{R \ln 2} = \frac{0.1}{10^5 \ln 2} = 1.4 \mu F \]

Problem 11
Consider a circular loop of current as shown in the figure, where the distance \( r \) to point P along the axis of the loop is much larger than the loop radius \( R \). If the distance \( r \) to point P is doubled, what is the ratio of the magnitude of the magnetic field at the new location of P to that at the original position?

(1) 1/8  (2) 8  (3) 1/4  (4) 4  (5) 1/2

For \( r \gg R \) the magnetic field of a current loop becomes that of a magnetic dipole with the following dependence:

\[ B \propto \frac{H_0 \mu}{r^3} \]

where \( \mu \) is the magnetic dipole moment iA.

Since this field falls off as \( r^{-3} \), when the distance is doubled, the field magnitude is 1/8 of the initial value.
Problem 12

In the figure shown, a metal rod is pulled at a constant velocity along two parallel metal rails, connected with a strip of metal at one end. A magnetic field of magnitude $B = 0.50\,\text{T}$ points out of the page. If the metal rod has a length $L = 0.25\,\text{m}$ and a resistance of $0.3\,\Omega$, what is the force necessary (in N) to pull the rod with a velocity of $2\,\text{m/s}$? The rails and connector have negligible resistance.

![Figure 12](image1.png)

(1) 0.1  (2) 0.25  (3) 0.8  (4) 0.03  (5) 2.0

The magnitude of the induced EMF in the loop is $\varepsilon = \nu LB$ from Faraday's law, since the amount of magnetic flux contained by the circuit is increasing. Since the flux is increasing, Lenz's Law says that the direction of the induced EMF and current will be to oppose that decrease, so the induced magnetic field should point into the page also. This requires a clockwise direction. The magnitude of the current in the rod is $i = \frac{\nu LB}{R}$

With a clockwise current, the current travels up the rod. This leads to a force pointing to the right: $\mathbf{F} = iL \times \mathbf{B}$, with magnitude:

$$F = \frac{\nu L^2 B^2}{R} = \frac{2(0.25)^2(0.5)^2}{0.3} = 0.10 \, \text{N}$$

Problem 13

The magnetic field passing perpendicularly through a square conducting loop of side length 2m as shown decreases linearly from $2\,\text{T}$ to 0 in 100 seconds. What is the induced EMF and the direction of the induced current?
(1) 0.08V, counter-clockwise  
(2) 0.08V, clockwise  
(3) 0.16V, counter-clockwise  
(4) 0.16V, clockwise  
(5) 0.02V, clockwise

Since the magnetic flux is decreasing, Lenz’s Law says the induced field will oppose that change. That leads to a counter-clockwise current in order to induce a field coming out of the page. By Faraday’s Law:

$$\varepsilon = \frac{d}{dt}(BA) = s^2 \frac{\Delta B}{\Delta t} = 2^2 \frac{2}{100} = 0.08V$$

**Problem 14**

The current through an inductor with inductance \(L=0.3H\) is shown by the graph, with the direction from left to right through the inductor as shown. What is the EMF across the inductor (\(V_L = V_{right} - V_{left}\), including sign, at \(t = 1\text{ms}\)?

![Graph showing current and voltage over time](graph.png)

(1) 900 V  
(2) -900 V  
(3) 1500 V  
(4) -750 V  
(5) -2 V

The induced EMF on an inductor is given by: \(\varepsilon = -L \frac{di}{dt}\)

At \(t=1\text{ms}\), \(\frac{di}{dt} = \frac{(2-8)}{(0.002-0)} = -\frac{6}{0.002} = -3000 \text{A/s}\)

So the EMF is \(V_L = -0.3(-3000) = 900 \text{ V}\)

**Problem 15**

The switch \(S\) in the shown RL circuit is closed at time \(t=0\). If \(L=10\text{H}, R=2 \Omega,\) and \(\varepsilon = 6V,\) what is the magnitude of the potential difference across the inductor immediately after closing the switch.
The current increases in an RL circuit as:

\[ i(t) = \frac{\varepsilon}{R} \left( 1 - e^{-\frac{t}{\tau_L}} \right) \]

where \( \tau_L = \frac{L}{R} \).

when \( t=0 \) then \( i=0 \), so there is no potential difference across the resistor. Thus by the loop rule the potential difference across the inductor must be equal to the EMF of 6V.

**Problem 16**

Two wires are made of the same material. The second wire has twice the radius and twice the length of the first wire. What is the ratio of the second wire’s resistance to the first one?

(1) ½  (2) 2  (3) 8  (4) 1/8  (5) 1

If the first wire has length \( L \) and radius \( r \), its resistance is

\[ R_1 = \rho L / (\pi r^2) \]

where \( \rho \) is the resistivity. Hence the resistance of the second wire is \( R_2 = \rho 2L / (4 \pi r^2) = R_1 / 2 \)

**Problem 17**

A 5 Ohm resistor dissipates 10 Watts of power. What is the current \( i \) (in amps) through the resistor and the voltage \( V \) (in volts) across it?

(1) \( i = \sqrt{2} \), \( V = 5 \sqrt{2} \)
(2) \( i = 5 \sqrt{2} \), \( V = \sqrt{2} \)
(3) \( i = 2 \), \( V = 5 \)
(4) \( i = 5 \), \( V = 2 \)
(5) \( i = \sqrt{10} \), \( V = \sqrt{10} \)
The power dissipated by a current $i$ flowing through a resistance $R$ is

$$P = i^2 R.$$ Hence the current is

$$i = \sqrt{\frac{P}{R}} = \sqrt{\frac{10 \text{ W}}{5 \Omega}} = \sqrt{2} \text{ A}.$$ The voltage across the same resistor is

$$V = iR = 5\sqrt{2} \text{ V}.$$

**Problem 18**

In the figure, $R = 14.0 \Omega$, $C = 6.20 \mu F$, and $L = 54.0 \text{ mH}$, and the ideal battery has emf $\varepsilon = 34.0 \text{ V}$. The switch is kept at $a$ for a long time and then thrown to position $b$. What are the frequency $f$ (not angular frequency!) in Hz and the amplitude of the current oscillation $I$ in Amps?

(1) $f = 275$, $i = .364$
(2) $f = 1730$, $i = .0580$
(3) $f = 43.8$, $i = 2.29$
(4) $f = 275$, $i = 2.43$
(5) $f = 1730$, $i = 2.43$

This was problem 31-17 from the assigned homework! Keeping the switch at $a$ for a long time fully charges the capacitor to charge $q_0 = CE = 210.8 \mu C$. When we then flip the switch to $b$ at time $t = 0$ then initial current is $i_0 = 0$ and the charge at any time is given by $q(t) = q_0 \cos(\omega t)$, where the angular frequency is $\omega = 1/\sqrt{LC}$. The frequency is therefore $f = \omega / 2\pi \sim 275 \text{ Hz}$. The current is $i(t) = dq / dt = -\omega q_0 \sin(\omega t)$, hence the amplitude of the current oscillation is $\omega q_0 \sim .364 \text{ A}$.

**Problem 19**

An LC circuit with capacitance $C = 4 \mu F$ and inductance $L = 2.5 \text{ mH}$ carries a total energy of $U = 10 \text{ mJ}$. At a particular instant the current through the inductor is $i = 2 \text{ A}$. What is the charge on the capacitor at the same instant?

(1) 200 $\mu C$  (2) 283 $\mu C$  (3) 400 $\mu C$  (4) 141 $\mu C$  (5) 566 $\mu C$

Recall that the energy stored in a capacitor is $U_c = q^2 / 2C$, and the energy stored in an inductor is $U_l = \frac{1}{2} Li^2$. Hence the charge on the capacitor is

$$q = \sqrt{2CU_c - CLi^2} = 2 \times 10^{-4} \text{ C}.$$
Problem 20

What resistance $R(\text{should be connected in series with an inductance } L = 220 \ \text{mH and capacitance } C = 12.0 \ \mu \text{F for the maximum charge on the capacitor to decay to 99.0\% of its initial value in 50.0 cycles? (Assume } \omega' \approx \omega.\text{)})$ 

1. 8.66 mΩ  
2. 4.33 mΩ  
3. 54.5 mΩ  
4. 3.76 mΩ  
5. 1.88 mΩ 

This was problem 31-25 from the assigned homework! Recall that the time evolutions of the capacitor charge in an RLC circuit takes the form 
$q(t) = Q e^{-Rt/2L} \cos(\omega't + \phi)$. The time for fifty cycles is 
$t = 50 \times 2\pi / \omega' = 100\pi \sqrt{LC}$. Having the charge decay to 99\% of its maximum value after 50 cycles means $e^{-Rt/2L} = .99$, which implies 
\[ R = \sqrt{LC} \times \ln(100 / 99) / 50\pi \approx 0.00866 \ \Omega. \]