Chapter 29  Magnetic Fields Due to Currents

In this chapter we will explore the relationship between an electric current and the magnetic field it generates in the space around it. We will follow a two-prong approach, depending on the symmetry of the problem.

For problems with low symmetry we will use the law of Biot-Savart in combination with the principle of superposition.

For problems with high symmetry we will introduce Ampere’s law.

Both approaches will be used to explore the magnetic field generated by currents in a variety of geometries (straight wire, wire loop, solenoid, toroid coil). We will also determine the force between two parallel current carrying conductors. We will then use this force to define the SI unit for electric current (the Ampere).

A proton (charge e), traveling perpendicular to a magnetic field, experiences the same force as an alpha particle (charge 2e) which is also traveling perpendicular to the same field. The ratio of their speeds, v_{proton}/v_{alpha}, is:

A. 0.5  B. 1  C. 2  D. 4  E. 8

A proton (charge e), traveling perpendicular to a magnetic field in a circular orbit, experiences the same force as an alpha particle (charge 2e, mass 4p) which is also traveling perpendicular to the same field. The ratio of their orbital frequency f_proton/f_alpha is:

A. 0.5  B. 1  C. 2  D. 4  E. 8

A proton (charge e), traveling perpendicular to a magnetic field in a circular orbit, experiences the same force as an alpha particle (charge 2e, mass 4p) which is also traveling perpendicular to the same field. The ratio of their orbital radii r_proton/r_alpha is:

A. 0.5  B. 1  C. 2  D. 4  E. 8

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The law of Biot-Savart

This law gives the magnetic field dB generated by a wire segment of length ds at a point in space. Consider the geometry shown in the figure. Associated with the wire segment of length ds is an associated vector dA that has magnitude which is equal to the length ds. The direction of dA is the same as that of the current that flows through the segment ds.

The magnetic field dB generated at point P by the element ds located at point A is given by the equation: dB = \( \mu_0 I ds \times \hat{r} / r^2 \). Here \( \hat{r} \) is the vector that connects point A (location of element ds) with point P at which we want to determine dB. The constant \( \mu_0 = 4\pi \times 10^{-7} \) T m/A and is known as "permeability constant". The magnitude of dB is:

\[ dB = \frac{\mu_0 I ds \sin \theta}{4\pi} \]

Here \( \theta \) is the angle between \( \hat{r} \) and \( \hat{A} \).

Consider the wire element of length ds shown in the figure. The element generates at point P a magnetic field of magnitude dB = \( \frac{\mu_0 I ds \sin \theta}{4\pi} \). Vector dB is pointing into the page. The magnetic field generated by the whole wire is found by integration.

\[ B = \frac{\mu_0 I}{2\pi R} \]

The magnitude of the magnetic field generated by the wire at point P located a distance R from the wire is given by the equation:

\[ B = \frac{\mu_0 I}{2\pi R} \]

The magnetic field lines form circles that have their centers at the wire. The magnetic field vector \( \mathbf{B} \) is tangent to the magnetic field lines. The sense for \( \mathbf{B} \) is given by the right hand rule. We point the thumb of the right hand in the direction of the current. The direction along which the fingers of the right hand curl around the wire gives the direction of \( \mathbf{B} \).
A wire section of length $ds$ generates at the center $C$ a magnetic field $d\vec{B}$

The magnitude $|d\vec{B}| = \frac{\mu_0 i ds \sin 90^\circ}{2\pi R} = \frac{\mu_0 i ds}{2\pi R}$

The length $ds = R d\phi$

$\rightarrow dB = \frac{\mu_0 i}{4\pi R} d\phi$

Vector $d\vec{B}$ points out of the page

The net magnetic field $\vec{B} = \int dB = \int \frac{\mu_0 i}{4\pi R} d\phi = \frac{\mu_0 i}{2\pi R}$

Note: The angle $\phi$ must be expressed in radians

For a circular wire $\phi = 2\pi$. In this case we get $B = \frac{\mu_0 i}{2\pi R}$ (29 - 5)

### Ampere's Law

The law of Biot-Savart combined with the principle of superposition can be used to determine $\vec{B}$ if we know the distribution of currents. In situations that have high symmetry we can use instead Ampere's law, because it is simpler to apply.

Ampere's law can be derived from the law of Biot-Savart with which it is mathematically equivalent. Ampere's law is more suitable for advances formulations of electromagnetism. It can be expressed as follows:

The line integral $\int \vec{B} \cdot d\vec{s}$ of the magnetic field $\vec{B}$ along any closed path is equal to the total current enclosed inside the path multiplied by $\mu_0$.

The closed path used is known an "Amperian loop". In its present form Ampere's law is not complete. A missing term was added by Clark Maxwell. The complete form of Ampere's law will be discussed in chapter 32.

### Implementation of Ampere's law:

1. Determination of $\int \vec{B} \cdot d\vec{s}$. The closed path is divided into $n$ elements $\Delta x_1, \Delta x_2, ..., \Delta x_n$. We then from the sum:

$$\sum_{i=1}^{n} B \Delta x_i = \sum_{i=1}^{n} B i \Delta\theta_i$$

Here $B_i$ is the magnetic field in the $i$-th element.

2. Calculation of $i_m$. We curl the fingers of the right hand in the direction in which the Amperian loop was traversed. We note the direction of the thumb.

All currents inside the loop parallel to the thumb are counted as positive. All currents inside the loop antiparallel to the thumb are counted as negative. All currents outside the loop are not counted.

In this example: $i_m = i_1 - i_2$ (29 - 8)

### Magnetic field outside a long straight wire

We already have seen that the magnetic field lines of the magnetic field generated by a long straight wire that carries a current $i$ have the form of circles which are concentric with the wire.

We choose an Amperian loop that reflects the cylindrical symmetry of the problem. The loop is also a circle of radius $r > R$ that has its center on the wire. The magnetic field is tangent to the loop and has a constant magnitude $B_i$.

$$\int B_i ds = \int B_i ds \cos 0 = B_i \int ds = 2\pi R B_i = \mu_0 i_m$$

$\rightarrow B_i = \frac{\mu_0 i_m}{2\pi R}$

Note: Ampere's law holds true for any closed path.

We choose to use the path that makes the calculation of $B_i$ as easy as possible.

(29 - 9)

### Magnetic field inside a long straight wire

We assume that the distribution of the current within the cross-section of the wire is uniform. The wire carries a current $i$ and has radius $R$.

We choose an Amperian loop is a circle of radius $r < R$ that has its center on the wire. The magnetic field is tangent to the loop and has a constant magnitude $B_i$.

$$\int B_i ds = \int B_i ds \cos 0 = B_i \int ds = 2\pi R B_i = \mu_0 i_m$$

$\rightarrow B_i = \frac{\mu_0 i_m}{2\pi R}$

$$2\pi R B_i = \mu_0 \frac{e^2}{2}\rightarrow B_i = \left(\frac{\mu_0}{2\pi R}\right)\frac{e^2}{2}$$ (29 - 10)
We will use Ampere's law to determine the magnetic field inside a solenoid. We assume that the magnetic field is uniform inside the solenoid and zero outside. We assume that the solenoid has \( n \) turns per unit length.

The magnetic field inside a toroid is not uniform. In vector form:

\[
\mathbf{B}(\zeta) = \frac{\mu_0 i R^2}{2\pi\zeta}
\]

The loop generates a magnetic field that has the same form as the field generated by a bar magnet.

The diagram shows a straight wire carrying current \( i \) in a uniform magnetic field. The magnetic force on the wire is indicated by an arrow but the magnetic field is not shown. Of the following possibilities, the direction of the magnetic field is:

A. opposite the direction of the current
B. opposite the direction of \(-\mathbf{F}\)
C. in the direction of \(-\mathbf{F}\)
D. into the page
E. out of the page

\[
\mathbf{B} = \mu_0 \mathbf{n} i
\]

\[
\int \mathbf{B} \cdot d\mathbf{l} = \int i ds = N_i \int ds = 2\pi N_i r
\]

An electron and a proton each travel with equal speeds around circular orbits in the same uniform magnetic field. The field is into the page. Because the electron is less massive than the proton and because the electron is negatively charged and the proton is positively charged:

A. the electron travels clockwise around the smaller circle and the proton travels clockwise around the larger circle
B. the electron travels counterclockwise around the smaller circle and the proton travels clockwise around the larger circle
C. the electron travels clockwise around the larger circle and the proton travels counterclockwise around the smaller circle
D. the electron travels counterclockwise around the smaller circle and the proton travels clockwise around the larger circle
E. the electron travels clockwise around the smaller circle and the proton travels counterclockwise around the larger circle

We will use the right hand rule. We curl the fingers of the right hand along the direction of the current in the coil windings. The thumb of the right hand points along \( \mathbf{B} \). The magnetic field inside the solenoid is parallel to the solenoid axis. The sense of \( \mathbf{B} \) can be determined using the right hand rule. We curl the fingers of the right hand along the direction of the current in the coil windings. The thumb of the right hand points along \( \mathbf{B} \). The magnetic field outside the solenoid is much weaker and can be taken to be approximately zero.

\[
\mathbf{B}(\zeta) = \frac{\mu_0 i R^2}{2\pi(\zeta + R)}
\]

The magnetic field generated by a wire coil of radius \( R \) which carries a current \( i \) is:

\[
B = \frac{\mu_0 i R^2}{2(\zeta + R)^2}
\]

For points far from the loop \( (\zeta \gg R) \) we can use the approximation:

\[
B = \frac{\mu_0 i R^2}{2\zeta^2}
\]

The enclosed current \( i_{enc} \) is:

\[
i_{enc} = n_i
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Note: The magnetic field inside a toroid is not uniform.

\[
\mathbf{B} = \frac{\mu_0 N_i}{2\pi r}
\]

A toroid has the shape of a doughnut (see figure). We assume that the toroid carries a current \( i \) and that it has \( N \) windings. The magnetic field lines inside the toroid form circles that are concentric with the toroid center. The magnetic field vector is tangent to these lines. The sense of \( \mathbf{B} \) can be found using the right hand rule. We curl the fingers of the right hand along the direction of the current in the coil windings. The thumb of the right hand points along \( \mathbf{B} \). The magnetic field outside the solenoid is much weaker and can be taken to be approximately zero.

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\[
\mathbf{B} = \frac{\mu_0 N_i}{2\pi r}
\]

The enclosed current \( i_{enc} \) is:

\[
i_{enc} = n_i
\]

Note: The magnetic field inside a toroid is not uniform.
The magnitude of the magnetic field at point \( P \), at the center of the semicircle shown, is given by:

A. \( \frac{2 \mu_0 i}{R} \)
B. \( \frac{\mu_0 i}{R} \)
C. \( \frac{\mu_0 i}{4\pi R} \)
D. \( \frac{\mu_0 i}{2R} \)
E. \( \frac{\mu_0 i}{4R} \)