Standing waves

\[ y_1 = A \sin(kx - \omega t) \]
\[ y_2 = A \sin(kx + \omega t) \]

**Superposition**

\[ y = A[\sin(kx - \omega t) + \sin(kx + \omega t)] \]

use same identity as before:

\[ y = 2A \sin \left( \frac{1}{2}(kx - \omega t + kx + \omega t) \right) \cos \left( \frac{1}{2}(kx - \omega t - kx + \omega t) \right) \]

\[ y = 2A \sin(kx) \cos(\omega t) \]

Not a traveling wave [which would be \( \sin(kx - \omega t) \)] but a standing wave.

The previous expression is the mathematical form of a standing wave.

A node (N) is a point of zero oscillation. An antinode (A) is a point of maximum displacement. All points between nodes oscillate up and down.

**Nodes and Antinodes**

The nodes occur where \( y(x,t) = 0 \).

\[ y(x,t) = 2A \cos \omega t \sin kx = 0 \]

The nodes are found from the locations where \( kx = 0, \pi, 2\pi, \ldots \)

\[ kx = n\pi \text{ and } n = 0, 1, 2, \ldots \]

The antinodes occur when \( \sin kx = \pm 1 \); that is where

\[ kx = \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \]

\[ kx = \frac{(2n+1)\pi}{2} \text{ and } n = 0, 1, 2, \ldots \]

**Standing waves on a string, both ends fixed**

If the string has a length \( L \), and both ends are fixed, then \( y(x = 0, t) = 0 \) and \( y(x = L, t) = 0 \) (node at both ends!!)

\[ y(x = 0, t) = \sin k(0) = 0 \]
\[ y(x = L, t) = \sin kL = 0 \]

The wavelength of a standing wave:

\[ \lambda = \frac{2\pi}{k} \]

\[ n \lambda = \frac{2\pi}{k} \lambda = \frac{2\pi}{k} = \frac{2\pi}{\lambda} = \frac{L}{n} \text{ where } n = 1, 2, 3, \ldots \]

These are the permitted wavelengths of standing waves on a string; no others are allowed.

The speed of the wave is:

\[ v = \lambda f \]

The allowed frequencies are then:

\[ f_n = \frac{v}{\lambda_n} = \frac{nv}{2L} \text{ for } n = 1, 2, 3, \ldots \]
Natural frequency and Resonance

The n = 1 frequency is called the fundamental frequency (n=2 first overtone, etc.).

\[ f_n = \frac{v}{\lambda} = \frac{nv}{2L} = n f_1 \]

All allowed frequencies (called harmonics) are integer multiples of \( f_1 \).

\[ n = 1 \] frequency is also called 1\(^{st}\) harmonic. \( n = 2 \) is called 2\(^{nd}\) harmonic, etc.

\[ \frac{\mu}{Fv} = \frac{F}{(\lambda f_1)^2} = \frac{F}{(62 \text{ N})^2} = \frac{F}{(329.63 \text{ Hz})^2 \times (0.65 \text{ m})^2} = 4.5 \times 10^{-4} \text{ kg/m} \]

Example Problem: Standing Waves

A string in a grand piano is 2 m long and has a mass density of 1 g/m. If the fundamental frequency of oscillations of the string is 440 Hz, what is the tension in the string (in N)?

Answer: 3097.6

\[ v = \sqrt{\frac{F}{\mu}} \]
\[ F = \mu v^2 \]
\[ F = 4 \mu f_1^2 = 4 \times 0.001 \text{ kg/m} \times (440 \text{ Hz})^2 \times (2 \text{ m}) \times (1000 \text{ s}^{-2}) \]
\[ = 3097.6 \text{ N} \]
\[ f_1 = \frac{v}{2L} \]
\[ v = 2L f_1 \]

Example Problem: Standing Waves

A nylon guitar string has a linear density of 5 g/m and is under a tension of 200 N. The fixed supports are D = 60 cm apart. The string is oscillating in the standing wave pattern shown in the figure. What is the frequency of the traveling waves whose superposition gives this standing wave?

Answer: 500 Hz

\[ D = \frac{\lambda}{2} \]
\[ \lambda = \frac{2D}{3} \]
\[ v = \frac{F}{\mu} \]
\[ f = \frac{v}{\lambda} = \frac{3}{2D} \]
\[ = \frac{3}{2 \times 0.6 \text{ m}} \times \frac{200 \text{ N}}{0.008 \text{ kg/m}} = 500 \text{ Hz} \]

Example Problem: Standing Waves

A string, which is tied to a sinusoidal oscillator at P and which runs over a support Q, is stretched by a block of mass m. The distance L = 2.0 m, the linear mass density of the string \( \mu = 4.9 \text{ g/m} \), and the oscillator frequency \( f = 100 \text{ Hz} \). The motion at P is in the vertical direction, and its amplitude is small enough for that point to be considered a node. A node also exists at Q. What mass allows the oscillator to set up the second harmonic on the string?

Answer: 20 kg

\[ m = \frac{2\mu f_1^2}{g} \]
\[ = \frac{2 \times 0.0049 \text{ kg/m} \times (4 \times 100 \text{ s}^{-2})^2}{9.8 \text{ m/s}^2} \]
\[ = 20 \text{ kg} \]
Example

Two sinusoidal waves, identical except for phase, travel in the same direction along a string and interfere to give a resultant wave

\[ y(x,t) = (3.0 \text{ mm}) \sin(20x - 4.0t + 0.820) \]

with \( x \) in meters and \( t \) in seconds.

What are a) wavelength; b) phase difference; c) amplitude of two original waves?

Example

Compare resultant

\[ y(x,t) = (3.0 \text{ mm}) \sin(20x - 4.0t + 0.820) \]

with general form for addition of two waves of same frequency & amplitude:

\[ y(x,t) = A \cos \left( \frac{\omega}{2} \right) \sin \left( kx - \omega t + \frac{1}{2} \phi \right) \]

Read off:

\[ 2A \cos(\phi/2) = 3.0 \text{ mm} \]
\[ \phi/2 = 0.820 \]
\[ k \equiv 2\pi/\lambda = 20 \text{ m}^{-1} \]
\[ \omega = 4.0 \text{s}^{-1} \]

which gives \( \lambda = 0.31 \text{ m}, \phi = 1.64 \text{ rad}, \Lambda = 2.2 \text{ mm} \)