\[ \alpha = 0.34 \times 10^2 \frac{m^2}{s} \]

Problem:
Suppose that you are installing a water pipe in a location where the air temperature in the winter can be \(-5^\circ C\) for time periods of up to 7 months. Find the minimum depth you must lay your pipe to ensure that it does not freeze, assuming the average soil temperature is initially \(20^\circ C\).

Solution:
Use eqn. \( \rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \)

Problems: This eqn is difficult to solve, as is, analytically, since it is a PDE. One common method to solve it is the method of similarity solutions.

This method is useful in this case because the system can be assumed semi-infinite and because we are able to combine the Initial condition with the B.C at \(\infty\) to a single boundary condition.

Application of Buckingham’s \( \pi \) Theorem:
Want: to use \( \pi \) to determine the minimum number of dimensionless variables needed to solve this problem which in turn implies that the minimum number of parameters are used.

Start:
\[ \rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \] \( \text{[1]} \)

Equation can be written as
\[ \rho c_p \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = 0 \]

I.C.: \( t = 0 \), \( T = T_i \)

\( \Rightarrow \) \( f( ) = 0 \)

B.C.: \( x = 0 \), \( T = T_{air} \)

\( \Rightarrow \) from our initial equation, I.C., and B.C.'s we find

\[ f( ) = f(T, T_i, T_{air}, c_p, k, T, x) = 0 \]
differential

Since this is a heat transfer problem, temperature is expressed as a difference and density, heat capacity, and thermal conductivity are combined to appear as \( \alpha = \frac{k}{\rho C_p} \). \( \alpha \) is known as the thermal diffusivity, which is analogous to the kinematic viscosity of fluids, and mass diffusivity.

\[
\Rightarrow f(T, T_i, T_{air}, \rho, C_p, k, x, t) = f(T_{air}, T_{air}, \alpha, x, t) = 0 \quad [27]
\]

Now our function consists of 5 physical quantities and 3 fundamental units (length, time, temperature). From B. T. T. there are at most 2 dimensionless variables needed for this problem. To find these variables, we will use B. T. T.

\[
f(T-T_{air}, T_{i}-T_{air}, \alpha, x, t) = 0
\]

can be re-written as

\[
T-T_{i} = \phi(T_{air}-T_{i}, \alpha, x, t)
\]

\[
\Rightarrow [T-T_{i}]=[T_{air}-T_{i}][\alpha]^{b}[x]^{c}[t]^{d}
\]

Using the Dorsin's relation

\[
\phi = [L]^{a}[T]^{b}[T]^{c}[L]^{d}
\]

Doing a balance on the 4 fundamental units:

Temp. (\( \phi \)): \( 1 = a \quad (a) \)

Time (\( T \)): \( 0 = c - b \quad (c) \)

Length (\( L \)): \( 0 = 2b + d \quad (d) \)

We now have 3 equations and 4 unknowns.

From (a), \( a = 1 \)

From (c), \( c = 6 \)

and from (d), \( d = -2b = -2c \)

Choosing \( d \) as the independent power we get,

\[
[T-T_{i}]=[T_{air}-T_{i}][\alpha]^{b}[x]^{c}[t]^{d} \quad \text{therefore}
\]
\[
\frac{[T - T_e]}{[T_w - T_e]} = \left[\frac{x}{\frac{T}{4x^2}}\right]^d
\]

Now
\[
\frac{[T - T_e]}{[T_w - T_e]} = 3\left(\frac{x}{T}\right)
\]

The choices for our dimensionless variables are
\[
\Theta = \frac{[T - T_e]}{[T_w - T_e]} \quad \text{and} \quad Z = \frac{x}{\frac{T}{4x^2}}
\]

Note: The factor of \(\frac{1}{4}\) in the denominator of \(Z\) is included because this is a well-studied problem and it leads to a neat solution of the original equation. Without prior knowledge of the solution, we would not have used this factor.

Solving the problem:

Now our original eqn. \(\rho C_p \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}\) or \(\frac{\partial^2 T}{\partial x^2} = \alpha \frac{\partial T}{\partial t}\)

By nondimensionalization becomes:
\[
\frac{\partial \Theta}{\partial Z} = -\frac{1}{\frac{n}{2}} \frac{d \Theta}{d Z} \quad \frac{\partial^2 \Theta}{\partial Z^2} = \frac{1}{4} \frac{d^2 \Theta}{d Z^2} \quad \frac{\partial^2 \Theta}{\partial Z^2} = 4 \frac{d \Theta}{d Z}
\]

Substituting these back into our equation we get

\[\text{Eq 3:} \quad \frac{d^2 \Theta}{d Z^2} + 2Z \frac{d \Theta}{d Z} = 0\]

Now the corresponding boundary and initial conditions become
\[
2=0, \quad \Theta = 1
\]
\[
2=\infty, \quad \Theta = 0
\]

One final substitution of \(Z = \frac{dT}{dZ}\) reduces Eq 3 into a 1st order separable equation.
Upon integration we get: 
\[ \Theta = C_1 \exp(-z^2) = \frac{d\Theta}{dz} \]
a second integration gives:
\[ \Theta = C_1 \int_0^z \exp(-z^2) \, dz + C_2 \]

After applying our boundary conditions \( C_2 = 1 \) and
\[ C_1 = \frac{1}{\int_0^\infty \exp(-z^2) \, dz} = -\frac{2}{\sqrt{\pi}} \]

\[ \Rightarrow \quad \Theta = 1 - \frac{2}{\sqrt{\pi}} \int_0^z \exp(-z^2) \, dz = 1 - \text{erf}(z) = \text{erfc}(z) \]

where \( \text{erfc} \) is the complementary \( \text{erf} \) function or
the gaussian distribution curve.

We can now use this relationship between

**temperature, depth, and time**

to solve our problem.

**We know** \( T_i = 20^\circ C \), \( T_{air} = -5^\circ C \), and \( T = 1^\circ C \)

\[ \Rightarrow \quad \Theta = \frac{T - T_i}{T_{air} - T_i} = \frac{-19^\circ C}{-28^\circ C} = 0.68 \]

\[ \therefore \text{ from graph } z = 0.68 = \frac{x}{\sqrt{4\alpha t}} \]

\[ t = 2 \text{ month} = 5.259498 \times 10^6 \text{ s} \]
\[ \alpha = 0.34 \times 10^{-6} \frac{\text{m}^2}{\text{s}} \]

\[ \Rightarrow x = 0.2\sqrt{4 \left(0.34 \times 10^{-6} \frac{\text{m}^2}{\text{s}} \right) \left(5.259498 \times 10^6 \text{ s} \right)} \]

\[ = 0.535 \text{ m} \]

\[ x \approx 1.75 \text{ ft} \]

This solution can analogously be applied to
momenum or diffusion.
Temp-Distribution in a semi-infinite slab

\[ \frac{(T-T_0)}{(T_{air}-T_0)} \]

against \[ \frac{x}{\sqrt{4a t}} \]