1.* (6.1)

On the surface of sphere (radius $R$)

$ds$: distance between two points $(0, \psi)$ and $(0+d\phi, \psi+d\psi)$

\[ ds = R d\phi \]

\[ ds = R \sin \psi d\phi \]

\[ ds = \sqrt{R^2 d\phi^2 + R^2 \sin^2 \psi d\phi^2} = R \sqrt{1 + \sin^2 \psi} d\phi \]

where \( \psi' = \frac{d\psi}{d\phi} \).

Total length between two points on the sphere

\[ L = R \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2 \psi} d\phi \]

\[ f(\psi, \psi', \theta) \]

(6.16)

\[ f(\psi, \psi', \theta) = \sqrt{1 + \sin^2 \psi} \]

Since \( \frac{df}{d\psi} = 0 \), \( \frac{df}{d\psi'} = 0 \)

Therefore, \( \frac{df}{d\psi'} = \frac{\sin \psi'}{\sqrt{1 + \sin^2 \psi}} = \text{const} = C \)

Since it has spherical symmetry, we can put point 1 at the north pole ($\theta = 0$). This makes $C = 0$.

Therefore for any angle $\theta$, $\psi'$ has to be 0. And $\psi = \text{const}$.

This means that the geodesics are always part of a great circle.
\[ (6.4) \]

\[
P_1 Q = \sqrt{x^2 + h_1^2 + z^2} \\
P_2 Q = \sqrt{(x_2-x)^2 + h_2^2 + z^2}
\]

Speed of light in a medium
\[ v = \frac{c}{n} \]

\[ T = \frac{P_1 G}{\left( \frac{c}{n_1} \right)} + \frac{P_2 Q}{\left( \frac{c}{n_2} \right)} \]

\[ = \frac{1}{c} \left( n_1 \sqrt{x^2 + h_1^2 + z^2} + n_2 \sqrt{(x_2-x)^2 + h_2^2 + z^2} \right) \]

Minimize \( T \) w.r.t \( z \) first.

\[
\frac{\partial T}{\partial z} = \frac{1}{c} \left\{ \frac{n_1 z}{\sqrt{x^2 + h_1^2 + z^2}} + \frac{n_2 z}{\sqrt{(x_2-x)^2 + h_2^2 + z^2}} \right\} = 0
\]

\[ z = 0 \]

\[
\frac{\partial T}{\partial x} = \frac{1}{c} \left\{ \frac{n_1 x}{\sqrt{x^2 + h_1^2}} + \frac{n_2 (x_2-x)}{\sqrt{(x_2-x)^2 + h_2^2}} \right\}
\]

\[ = \frac{1}{c} \left\{ n_1 \sin \theta_1 - n_1 \sin \theta_2 \right\} = 0 \]

\[ \therefore \quad n_1 \sin \theta_1 = n_2 \sin \theta_2 \quad \text{(Snell's Law)} \]

3. \( (6.11) \)

\[
\int_0^{x_2} \sqrt{x_1} \sqrt{1+y^2} \, dx : \quad f(y, y' \mid x) = \sqrt{x_1} \sqrt{1+y^2}
\]

\[ \frac{\partial f}{\partial y} = 0 ; \quad \frac{d}{dx} \left( \frac{\sqrt{x_1} \sqrt{y^2}}{\sqrt{1+y^2}} \right) = 0 \]
\[ \frac{\sqrt{2}y'}{\sqrt{1+y'^2}} = C \quad \text{(const.)} \]
\[ y' = \frac{C}{\sqrt{x-x^2}} \quad \therefore y = \int \frac{C}{\sqrt{x-x^2}} = 2C\sqrt{x-x^2} + A \]
\[ \text{or: } x = \frac{1}{4c^2}(y-A)^2 + c^2 \quad \text{int. const.} \]

4. [6.12] \[ \int_{x_1}^{x} x\sqrt{1-y'^2} \, dx : \quad f(x, y, x) = x\sqrt{1-y'^2} \]
\[ \frac{dF}{dy} = 0, \quad \frac{df}{dx}(x, y') = \frac{-xy'}{\sqrt{1-y'^2}} = 0 \]
\[ \frac{xy'}{\sqrt{1-y'^2}} = C \quad \text{(const.)} \]
\[ y' = \frac{C}{\sqrt{x^2+c^2}} \]
\[ y = \int \frac{C}{\sqrt{x^2+c^2}} \, dx = \frac{x}{c} = \sinh z \]
\[ \frac{C}{c} + A = C\sinh \left( \frac{z}{c} \right) + A \]
\[ \text{or } x = C\sinh \left( \frac{y-A}{c} \right) \]
(a) \( x = a(\theta - \sin \theta) = \pi b \rightarrow a = b \) \& \( \theta_2 = \pi \).
\( y = a(1 - \cos \theta) = 2b \)
\[ t = \int_1^\pi \frac{d\theta}{\sqrt{\nu^2 (x^2 + y^2)}} \]
\[ ds = \sqrt{(dx)^2 + (dy)^2} = d\theta \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} = \sqrt{x^2 + y^2} \, d\theta. \]
\[ t = \frac{\sqrt{b}}{\sqrt{2g}} \int_0^\pi \frac{\sqrt{(1-\cos \theta)^2 + \sin^2 \theta}}{\sqrt{1-\cos \theta}} \, d\theta \]
\[ \frac{dx}{d\theta} = b(1-\cos \theta) \]
\[ \frac{dy}{d\theta} = b\sin \theta \]
\[ \therefore t = \pi \frac{\sqrt{b}}{\sqrt{2g}} \int_0^\pi \frac{\sqrt{1-\cos \theta}}{\sqrt{1-\cos \theta}} \, d\theta \]
\[ = \pi \sqrt{\frac{b}{g}}. \]
\( x = a(\theta - \sin \theta) = 2\pi b \rightarrow a = b, \theta_2 = 2\pi. \)
\( y = a(1 - \cos \theta) = 0 \)

Similarly, \( t = 2\pi \sqrt{\frac{b}{g}}. \)
6. At point \((x, y)\)

(a) \[ V_x = V_0 \cos \phi + V_y \]
\[ V_y = V_0 \sin \phi \]

\[ V = \sqrt{(V_0 \cos \phi + V_y)^2 + V_0^2 \sin^2 \phi} \]
\[ = \sqrt{V_0^2 + 2V_0 V_y + V_y^2} \]

for \(y \ll 1\), \[ V \approx V_0 + V_y = V_0 (1 + ky) \; ; \; k = \frac{V}{V_0} \]

(b) \[ t = \int_0^P \frac{ds}{V} = \int_0^P \frac{\sqrt{1+y'^2}}{V_0 (1 + ky)} \, dx < \int_0^P \frac{1 + \frac{1}{2}y'^2}{V_0 (1 + ky)} \, dx \]
\[ = \frac{1}{V_0} \int_D^P \frac{f(y, y', x)}{1 + ky} \, dx \]
\[ f(y, y', x) = \frac{1 + \frac{1}{2}y'^2}{1 + ky} \]

(c) \[ \frac{2f}{\partial y} = k \left(1 + \frac{1}{2}y'^2 \right) \]
\[ \frac{2f}{\partial y'} = \frac{y'}{(1 + ky)^2} : \quad \frac{d}{dx} \left( \frac{2f}{\partial y'} \right) = \frac{(1+ky)y'' - ky'^2}{(1+ky)^2} \]
\[ \frac{2f}{\partial y} \frac{d}{dx} \left( \frac{2f}{\partial y'} \right) = -\frac{1}{(1+ky)^2} \left( (1+ky)y'' - \frac{1}{2}ky'^2 + k \right) = 0. \]

\[ (1+ky)y'' - \frac{1}{2}ky'^2 + k = 0 \]

Try \( y = \lambda (D-x) \), \[ y' = \lambda (D-2x) \], \[ y'' = -2\lambda \]
\[ (1+ky)(D-x) \left( \frac{-2\lambda}{1+ky} \right) - \frac{1}{2}k\lambda (D-2x)^2 + k \]
\[ = -2\lambda - 2k\lambda dx + 2k\lambda^2x - \frac{1}{2}k\lambda^2 x^2 + 2k\lambda^2 (D-2x)^2 + k \]

\[ = \frac{2k\lambda x^2 - 4k\lambda x + 2k\lambda x}{(1+ky)^2} \]
\[ -2\lambda - \frac{1}{2} k \lambda^2 \partial^2 + 1 = 0. \]

\[ \therefore 1k \partial^2 \lambda^2 + 4 \lambda - 2k = 0 \]

\[ \therefore \lambda = \frac{-2 + \sqrt{4 + 2k^2 \partial^2}}{4k \partial^2} \quad \text{(only positive soln.)} \]

7. (6.25).
\[ \gamma = a(\theta - \sin \theta) \]
\[ y = a(1 - \cos \theta) \]

From Prob. 5(c). \[ ds = \sqrt{x^2 + y^2} \, d\theta \]
\[ = a \sqrt{2(1 - \cos \theta)} \, d\theta. \]

The can is at rest at \( y = y_0 \).

Therefore \[ U = \sqrt{2g(y_0 + y)} = \sqrt{2g(\cos \theta_0 - \cos \theta)} \]

\[ \therefore t = \int_{\theta_0}^{\theta} \frac{ds}{U} = \int_{\theta_0}^{\pi} \frac{a \sqrt{2(1 - \cos \theta)}}{\sqrt{2g} \sqrt{\cos \theta_0 - \cos \theta}} \, d\theta. \]

\[ \theta = \pi - 2\alpha \quad \rightarrow \quad d\theta = -2d\alpha \]

Then,
\[ t = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\theta} \frac{\sqrt{2 \cos \theta (1 - 2\alpha)}}{\sqrt{2(\sin^2 \theta_0 - \sin^2 \theta)}} \, d\theta. \]

\[ \cos(\pi - 2\alpha) = \cos \pi \cos 2\alpha + \sin \pi \sin 2\alpha = -\cos 2\alpha \]
\[ = \sin^2 \alpha - \cos^2 \alpha = 1 - 2\cos^2 \alpha \]
\[ = 2\sin^2 \alpha - 1 \]
\[ t = 2 \sqrt{\frac{a}{g}} \int_0^{\theta_0} \frac{\cos \alpha \, d\alpha}{\sqrt{\sin^2 \alpha_0 - \sin^2 \alpha}} = 2 \sqrt{\frac{a}{g}} \int_0^{\theta_0} \frac{du}{\sqrt{u_0^2 - u^2}} \]

\[ \sin \alpha = u \quad \cos \alpha \, du = du \]

Again, \( \frac{u}{u_0} = x \quad \frac{du}{u_0} = dx \)

\[ t = 2 \sqrt{\frac{a}{g}} \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \pi \sqrt{\frac{a}{g}} \]

\[ \cos \theta = x \quad \sin \theta \, d\theta = dx \]

No \( t \) dependence.