Relativity

Lorentz transformation

A coordinate rotation:

\[ t' = t \]
\[ x' = x \cos \theta + y \sin \theta \]
\[ y' = -x \sin \theta + y \cos \theta \]
\[ z' = z' \]

Notice that distance is conserved under a coordinate rotation:

\[ x'^2 + y'^2 = (x \cos \theta + y \sin \theta)^2 + (-x \sin \theta + y \cos \theta)^2 \]
\[ = x^2 \cos^2 \theta + 2xy \cos \theta \sin \theta + y^2 \sin^2 \theta + x^2 \sin^2 \theta + x^2 \sin^2 \theta - 2xy \cos \theta \sin \theta + y^2 \cos^2 \theta \]
\[ = x^2(\cos^2 \theta + \sin^2 \theta) + y^2(\sin^2 \theta + \cos \theta) \]
\[ = x^2 + y^2. \]

This might seem obvious because the mathematics just confirms your everyday experiences.

The Galilean transformation:

\[ t' = t \]
\[ x' = x - vt \]
\[ y' = y \]
\[ z' = z \]

Inverse transformation:

\[ t = t' \]
\[ x = x' + vt' \]
\[ y = y' \]
\[ z = z' \]

The Lorentz transformation:

\[ t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \]
\[ x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \]
\[ y' = y \]
\[ z' = z \]
Inverse transformation:

\[
\begin{align*}
    t &= t' + \frac{vx'}{c^2} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \\
    x &= x' + vt' \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \\
    y &= y' \\
    z &= z'
\end{align*}
\]

Notice that in the limit that \( v/c \to 0 \), but \( v \) remains finite, the Lorentz transformations approach the Galilean transformation. So, only when \( v \) is comparable to \( c \) are the effects of special relativity revealed.

**Derive time dilation from the Lorentz transformations:**

Two events, \#1 at \((t_1, x_1)\) and \#2 at \((t_2, x_2)\), with occur at the same place \((x_1 = x_2)\) in the \( t, x \) coordinate system. Thus the proper time between the events is

\[
\Delta T_0 = t_2 - t_1.
\]

The time between the events in the primed coordinate system is

\[
\Delta T' = t'_2 - t'_1 = \frac{t_2 - vx_2/c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{t_1 - vx_1/c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{t_2 - vx_2/c^2 - t_1 + vx_1/c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{t_2 - t_1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{because} \quad x_1 = x_2 = \Delta T_0 \sqrt{1 - \frac{v^2}{c^2}},
\]

which is the correct expression for time dilation.

**Derive length contraction from the Lorentz transformations:**

A stick is at rest in the unprimed \( t, x \) coordinate system. Two events, \#1 at \((t_1, x_1)\) and \#2 at \((t_2, x_2)\), occur at different times \( t_1 \neq t_2 \) but at either end of the stick. And because the stick is at rest in the unprimed coordinate system, the proper length of the stick is

\[
L_0 = x_2 - x_1
\]

It has been arranged that these same two events occur at the same time in the primed coordinate system, \( t'_1 = t'_2 \). So, in the primed coordinates the length of the stick is measured
to be \( L' = x'_2 - x'_1 \). Now, the proper length of the stick is

\[
L_0 = x_2 - x_1 = \frac{x'_2 - vt'_2}{\sqrt{1 - v^2/c^2}} - \frac{x'_1 - vt'_1}{\sqrt{1 - v^2/c^2}} = \frac{x'_2 - vt'_2 - x'_1 + vt'_1}{\sqrt{1 - v^2/c^2}} = \frac{x'_2 - x'_1}{\sqrt{1 - v^2/c^2}} \quad \text{because } t'_1 = t'_2
\]

\[
L' = \frac{L'}{\sqrt{1 - v^2/c^2}} = \frac{L'_0}{\sqrt{1 - v^2/c^2}}
\]

which is the correct expression for length contraction.

**Consider the Galilean addition of velocities:**

With the Lorentz transformations in hand, we can now see how velocities are viewed from different coordinate systems. First consider the Galilean addition of velocities. Consider a bird flying along, and note two nearby events along the bird’s path—perhaps the events are two flaps of the bird’s wings. These events are noted to occur at \((t_1, x_1, y_1)\) and \((t_2, x_2, y_2)\) in the unprimed coordinate system, and at \((t'_1, x'_1, y'_1)\) and \((t'_2, x'_2, y'_2)\) in the primed coordinate system. The components of the speed of the bird in the unprimed system are

\[
u_x = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t} \quad \text{and} \quad \nu_y = \frac{y_2 - y_1}{t_2 - t_1} = \frac{\Delta y}{\Delta t}
\]

and in the primed coordinates

\[
u'_x = \frac{x'_2 - x'_1}{t'_2 - t'_1} = \frac{\Delta x'}{\Delta t'} \quad \text{and} \quad \nu'_y = \frac{y'_2 - y'_1}{t'_2 - t'_1} = \frac{\Delta y'}{\Delta t'}
\]

Now use the Lorentz transformations to relate these different speeds:

\[
u_x = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t} = \frac{(\Delta x' + v\Delta t')/\sqrt{1 - v^2/c^2}}{\sqrt{1 - v^2/c^2}} = \frac{\Delta x'/\Delta t' + v}{\sqrt{1 + v\Delta x'/\Delta t'c^2}} \quad \text{and, finally}
\]

\[
u_x = \frac{u'_x + v}{\sqrt{1 + vu'_x/c^2}}.
\]
and

\[ u_y = \frac{y_2 - y_1}{t_2 - t_1} = \frac{\Delta y}{\Delta t} = \frac{\Delta y'}{(\Delta t' + v\Delta x'/c^2)/\sqrt{1 - v^2/c^2}} = \frac{\Delta y'/\Delta t'}{(1 + v\Delta x'/\Delta t'c^2)/\sqrt{1 - v^2/c^2}} \] and, finally

\[ u_y = \frac{u'_y\sqrt{1 - v^2/c^2}}{1 + vu'_x/c^2}. \]

If \( v \) and \( u'_x \) and \( u'_y \) are all less than \( c \) then \( u_x, u_y, \) and also \( \sqrt{u'^2_x + u'^2_y} \) are all less than \( c \). Try this out with \( v = u'_x = u'_y = 9c/10 \).