Due at the start of class on Friday, January 17. Half credit will be available for homework submitted after the deadline but no later than the start of class on Wednesday, January 22.

Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. Consider the Euler-Lagrange equation for \( f = f(x(t), \dot{x}(t), t) : \)

\[
\frac{\partial f}{\partial x} - \frac{d}{dt} \left( \frac{\partial f}{\partial \dot{x}} \right) = 0.
\]

In class we saw that, if \( \frac{\partial f}{\partial x} = 0 \), then \( \frac{\partial f}{\partial \dot{x}} = \text{constant} \):

a) Show that, if \( \frac{\partial f}{\partial t} = 0 \), i.e., \( f \) is independent of its third argument, then:

\[
\frac{df}{dt} = \frac{\partial f}{\partial x} \dot{x} + \frac{\partial f}{\partial \dot{x}} \ddot{x}.
\]

b) By using the Euler-Lagrange equation above, eliminate \( \frac{\partial f}{\partial x} \) to show that * can be written as:

\[
\frac{df}{dt} = \frac{d}{dt} \left( \dot{x} \frac{\partial f}{\partial \dot{x}} \right),
\]

and integrate this to show that \( f - \dot{x} \frac{\partial f}{\partial \dot{x}} \) is a constant.

2. Consider the problem of a particle falling under gravity along a frictionless wire in the \( xy- \)plane. Take \( y \) to be positive in the downward direction.

a) Show that the velocity at any point \((x, y)\) along the path can be given by \( v = \sqrt{2gy} \).

b) Show that the time take to travel from point \( A \) to point \( B \) can be given by:

\[
T = \int_A^B dt \frac{\sqrt{x_2^2 + y_2^2}}{v} = \int_A^B dx \frac{\sqrt{1 + (dy/dx)^2}}{\sqrt{2gy}} = \int_A^B dy \frac{\sqrt{1 + (dx/dy)^2}}{\sqrt{2gy}}.
\]

c) Using one of these forms, write down the resulting Euler equation(s) for the motion.

d) Solve your result from part c). Hint: It may be best to express \( x \) and \( y \) in terms of some angular parameter: e.g., \( y = y_0 \sin^2(\theta/2) = y_0(1 - \cos(\theta))/2 \). This gives a parametric form for the shape of the wire which results in the shortest travel time.

e) Show that the shortest time to travel, while entirely under the influence of gravity, between two points a distance \( X \) apart, both located at \( y = 0 \), is \( T = \sqrt{2\pi X/g} \).
3. Two particles, of mass \( m_1 \) and \( m_2 \) respectively, are confined to the \( x \)-axis and are connected by a spring with potential energy \( U = \frac{1}{2}kx^2 \), where \( x = x_1 - x_2 - l \) is the extension of the spring and \( l \) is its un-stretched length.

   a) Write down the Lagrangian \( L(x_1, x_2, \dot{x}_1, \dot{x}_2) \) for the two particles.

   b) Rewrite \( L \) in terms of \( X_{CM} \), the center of mass position of the two masses, and \( x \), the extension of the spring.

   c) Write down the two Euler-Lagrangian equations, for \( X_{CM} \) and \( x \) respectively.

   d) Solve for \( X_{CM}(t) \) and \( x(t) \) and describe the motion of each.

4. Consider a double Atwood machine constructed as follows. A mass \( 4m \) is suspended from a massless string which passes over a massless pulley on frictionless bearings. The other end of this string supports a second similar pulley, over which passes a second massless string supporting a mass of \( 3m \) at one end and \( m \) at the other.

   a) Using two suitable, generalized coordinates, set up an unconstrained Lagrangian to describe the system.

   b) Write down the Euler-Lagrange equations for the two degrees of freedom.

   c) Find the acceleration of each of the masses when the system is released.

   d) Explain why the top pulley rotates while carrying equal weights on both sides.

5. Consider two arbitrary (non-diagonal) points on a sphere, given by \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) respectively.

   a) From purely geometric considerations, write down the equations for a point \((x, y, z)\) which lies on the great circle between them.

   b) By converting to polar coordinates throughout, rewrite this result as a single equation and, by suitable substitutions, express it in terms of the extremal values of \( \theta_{\text{max}} \) or \( \theta_{\text{min}} \) and an angle \( \phi_0 \) at which the great circle crosses the equator.

   c) Write down an extremal principle for the geodesic (shortest path) on a sphere and the Euler equation(s) which follow form it.

   d) Solve the Euler equation(s) completely and show that your solution is consistent with your result in part b). \( \text{Hint:} \) You may find it helpful to make several trigonometric substitutions. If all else fails, show that your answer in part b) satisfies the Euler equation(s) you obtained in part c).