PH4222, Section 3801, Spring 2014, Homework 12

Due at the start of class on Friday, April 18. Half credit will be available for late homework submitted no later than the start of class on Monday, April 21.

Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. An electron and a positron are traveling towards each other.
   a) In the center of mass frame, each has $|\mathbf{v}| = 0.75c$. What is their relative velocity toward each other.
   b) What is the energy of each photon emitted in the CM frame?
   c) In the lab frame the electron is at rest. What Lorentz transformation is required to bring the electron to rest?
   d) In the lab frame, what is the energy of each of the emitted photons?

2. This question involves working with space-time tensors. These obey similar transformation laws to co- and contra-variant vectors, but may have more (or fewer) indices.
   a) Let:
      \[
      x^\mu = \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}
      \]
      be the 4-position of a particle and:
      \[
      V^\mu = \frac{dx^\mu}{d\tau} = \begin{pmatrix} \frac{dx}{d\tau} \\ \frac{dy}{d\tau} \\ \frac{dz}{d\tau} \\ \frac{c dt}{d\tau} \end{pmatrix},
      \]
      its 4-velocity, where $\tau$ is proper time along its world-line. Prove that $V^\mu$ transforms correctly as a contra-variant (equivalent to a column-like) vector, i.e., that:
      \[
      V'^\mu = \frac{dx'^\mu}{d\tau} = \frac{\partial x'^\mu}{\partial x^\nu} \frac{dx^\nu}{d\tau}.
      \]
   b) Let:
      \[
      \nabla_\mu f(x, y, z, ct) \equiv \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial ct} \right)
      \]
      Prove that this transforms as a space-time covariant (that is, row-like) vector, i.e., that:
      \[
      \nabla'_\mu f(x', y', z', ct') = \frac{\partial x'^\mu}{\partial x^\nu} \nabla_\nu f(x, y, z, ct)
      \]
c) For a fixed world-line, prove that $V^\mu \nabla_\mu f$ is a space-time scalar, i.e., show that it is invariant under a coordinate transformation. Note: you will need to use:

$$\frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu, \quad \text{or} \quad \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x'^\mu}{\partial x^\nu} = \delta'^\mu_\mu$$

where $\delta^\nu_\mu$ (and $\delta'^\mu_\mu$) is the unit matrix.

d) Let $V^\mu$ and $U^\nu$ be two different space-time vectors. Prove that $T^\mu\nu = V^\mu U^\nu$ transforms correctly as a contra-variant space-time tensor of rank 2.

e) The entity:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

is a space-time scalar, because the metric $g_{\mu\nu}$ is a covariant space-time tensor of rank 2. How does $g_{\mu\nu}$ transform under a coordinate transformation?

f) What sort of a space-time quantity (tensor, vector, scalar) is $V^\mu U^\nu g_{\mu\nu}$? How does it transform under a coordinate transformation? How does $U^\nu g_{\mu\nu}$ (often denoted $U^\mu$ for convenience) transform?

3. One way to define the electromagnetic field tensor is by:

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu,$$

where $A_\mu = (A, -\phi/c)$ and $A = A_i$ is a covariant 3-vector.

a) If $E = -\nabla \phi - dA/dt$ and $B = \nabla \times A$, show that

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & E_1/c \\ -B_3 & 0 & B_1 & E_2/c \\ B_2 & -B_1 & 0 & E_3/c \\ -E_1/c & -E_2/c & -E_3/c & 0 \end{pmatrix}.$$ 

b) Consider the Lorentz transformation:

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & 1 & -\gamma \beta \\ -\gamma \beta & 1 & \gamma \\ 1 & \gamma & \gamma \beta \end{pmatrix} \quad \text{and its inverse } \Lambda^\mu_\nu = \begin{pmatrix} \gamma & 1 & \gamma \beta \\ -\gamma \beta & 1 & \gamma \\ \gamma & \gamma \beta & \gamma \end{pmatrix},$$

and construct

$$F'_{\mu'\nu'} = F_{\mu\nu} \Lambda^\mu_\mu \Lambda^\nu_\nu.$$ 

c) Show that your result in b) is equivalent to:

$$E'_1 = E_1, \quad E'_2 = \gamma(E_2 - \beta cB_3), \quad E'_3 = \gamma(E_3 + \beta cB_2), \quad \text{and}$$

$$B'_1 = B_1, \quad B'_2 = \gamma(B_2 + \beta E_3/c), \quad B'_3 = \gamma(B_3 - \beta E_2/c).$$
d) Hence prove that $\mathbf{E} \cdot \mathbf{B}$ and $\mathbf{E}^2 - c^2 \mathbf{B}^2$ are both invariant under Lorentz transformations.

e) Thus prove that if $\mathbf{E}$ and $\mathbf{B}$ are perpendicular in one inertial frame $S$, they are perpendicular in all other inertial frames $S'$.

f) Similarly, show that if $E > cB$ in frame $S$, there cannot exist a frame in which $E = 0$.

4. a) Show that if we make the change of variables $\xi = x - ct$ and $\eta = x + ct$ then:

$$
\frac{\partial^2 u(x, t)}{\partial t^2} - c^2 \frac{\partial^2 u(x, t)}{\partial x^2} = -4c^2 \frac{\partial}{\partial \xi} \frac{\partial u(x(\xi, \eta), t(\xi, \eta))}{\partial \eta},
$$

and hence write down a complete solution for a wave on an infinite string.

b) Consider a string of length $L$ and clamped at both ends. At $t = 0$, suppose that the displacement of the string is given by $u(x, 0) = u_0 \sin(n\pi x/L)$ where $n$ is an integer and its instantaneous velocity satisfies $\dot{u}(x, 0) = 0$. By the separation of variables (or some other approach) find the full solution describing the motion of the string.

c) Given that:

$$\nabla^2 f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial f}{\partial x^j} \right)$$

show that, in spherical polar coordinates (with $g_{ij} = \text{diag}(1, r^2, r^2 \sin^2 \theta)$), this is equivalent to:

$$\nabla^2 f = \frac{1}{r} \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}.$$  

d) Since waves in three dimensions obey:

$$\frac{\partial^2 f(x^i, t)}{\partial t^2} - c^2 \nabla^2 f(x^i, t) = 0,$$

show that a spherical light wave (away from the origin $r = 0$) can be given by:

$$f(x^i, t) = \Re \left( \frac{F_0 \exp i(kr \pm \omega t)}{r} \right),$$

where $f$ is any component of $\mathbf{A}$, and find the relation between $k$ and $\omega$.

e) There can be an infinite number of such solutions. What quantities can be varied in creating sums of such solutions? Argue that, if the integral exists and is sufficiently differentiable, then:

$$f(x^i, t) = \Re \int_{-\infty}^{\infty} \frac{F(\omega) \exp i(k(\omega)r - \omega t)}{r} d\omega,$$

is also a solution (away from $r = 0$).