(1) Suppose $\vec{J}(\vec{r})$ is constant in time but $\rho(\vec{r}, t)$ is not — conditions that might prevail, for instance, during the charging of a capacitor.

a) Show that the charge density at any particular point is a linear function of time:

$$\rho(\vec{r}, t) = \rho(\vec{r}, 0) + \dot{\rho}(\vec{r}, 0)t,$$

where $\dot{\rho}(\vec{r}, 0)$ is the time derivative of $\rho$ at $t = 0$. **(16 points)**

b) Suppose that the magnetic field is given at any time by the Biot-Savart law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{\|\vec{r} - \vec{r}'\|^3}.$$  

Check that this obeys the Maxwell equation $\nabla \cdot \vec{B} = 0$. **(16 points)**

c) Use the magnetic field given above, along with Faraday’s law, to show that the electric field is the gradient of a scalar potential. **(16 points)**

d) Find an integral expression for the electric field like the one given above for the magnetic field. **(16 points)**

e) Show that fields obey the Ampere/Maxwell law. **(16 points)**

(2) Picture the electron as a uniformly charged spherical shell, with charge $e$ and radius $R$, spinning at angular velocity $\omega \hat{z}$. Recall that the electric and magnetic fields are:

$$\vec{E} = \frac{e}{4\pi\varepsilon_0 r^2} \hat{r} \sin \theta \left[ \cos(kr - \omega t) - \sin(kr - \omega t) \frac{kr}{k^2} \right], \quad \vec{B} = \frac{2}{3} \mu_0 \sigma \omega R \hat{z} \theta(R - r) + \mu_0 \sigma \omega \frac{e}{3r^3} \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right) \theta(r - R).$$

a) What is the total energy $U$ contained in electromagnetic fields? **(10 points)**

b) What is the total angular momentum $\vec{L}$ contained in the fields? **(10 points)**

c) Suppose we set the energy and the angular momentum to their measured values:

$$U = m_e c^2, \quad \vec{L} = \frac{1}{2} \hbar \hat{z}.$$  

Express $R$ and $\omega$ in terms of the fine structure constant $\alpha \equiv e^2/(4\pi\varepsilon_0 \hbar c)$ and the Compton wavelength of the electron $\lambda_C \equiv \hbar/(m_e c)$. **(10 points)**

(3) Suppose

$$\vec{E}(r, \theta, \phi, t) = A \sin \theta \left[ \cos(kr - \omega t) - \frac{\sin(kr - \omega t)}{kr^2} \right] \hat{\phi}, \quad \text{with} \quad \frac{k^2}{r} = c.$$  

a) What is the associated magnetic field $\vec{B}(r, \theta, \phi, t)$? **(20 points)**

b) Show that these fields obey Maxwell’s equations in vacuum. **(20 points)**

c) Calculate the Poynting vector $\vec{S}(r, \theta, \phi, t)$. **(20 points)**

d) Integrate $\vec{S} \cdot d\vec{a}$ over a spherical surface. **(15 points)**