Vector space:

Addition and scalar multiplication
so that if \( \mathbf{v}_1 \) \& \( \mathbf{v}_2 \) are vectors, so is \( c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 \).

Associativity of addition: \( \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3) = (\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 \)

Commutativity of addition: \( \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1 \)

Zero vector: \( 0 + \mathbf{v} = \mathbf{v} + 0 \) for any \( \mathbf{v} \)

Inverse: \( \mathbf{v} + (-\mathbf{v}) = (-\mathbf{v}) + \mathbf{v} = 0 \)

Distributivity of scalar multiplication:
\( c(\mathbf{v}_1 + \mathbf{v}_2) = c\mathbf{v}_1 + c\mathbf{v}_2 \)
also \( (c_1 + c_2)\mathbf{v} = c_1\mathbf{v} + c_2\mathbf{v} \)
and \( c_1(c_2\mathbf{v}) = (c_1 c_2)\mathbf{v} \)

Identity under scalar multiplication: \( 1 \mathbf{v} = \mathbf{v} \)

Wavefunctions, \( \Psi(x) \), form a vector space with the scalars being complex.
The zero vector is the function which is zero everywhere.

In many cases we will denote the vectors as \( |\Psi\rangle \).
Inner product:

In 3D $\vec{a} \cdot \vec{b} = |\vec{a}|^2 \geq 0$
and $\vec{a} \cdot \vec{b} = 0$ iff $\vec{b} = \vec{0}$

For the function vector space the inner product is

$$\langle f | g \rangle = \int_\mathbb{R} f^*(x) g(x) \, dx.$$  

Note that $\langle f | f \rangle = \int_\mathbb{R} |f(x)|^2 \geq 0$
and $\int_\mathbb{R} |f(x)|^2 = 0$ iff $f(x) = 0$.

Properties:

$$\langle f | g_1 + g_2 \rangle = \langle f | g_1 \rangle + \langle f | g_2 \rangle$$
$$\langle f_1 + f_2 | g \rangle = \langle f_1 | g \rangle + \langle f_2 | g \rangle$$
$$\langle f | c g \rangle = c \langle f | g \rangle$$
$$\langle c f | g \rangle = c^* \langle f | g \rangle$$
$$\langle f | g \rangle^* = \langle g | f \rangle$$
Basis:

In 2D $\vec{V} = c_1 \hat{x} + c_2 \hat{y}$, where

$c_1 = \hat{x} \cdot \vec{V}$
$c_2 = \hat{y} \cdot \vec{V}$

and $\hat{x} \cdot \hat{x} = 1$, $\hat{y} \cdot \hat{y} = 1$, $\hat{x} \cdot \hat{y} = 0$.

(orthonormal basis)

Consider either the eigenstates of the infinite square well ($0 < x < a$ and $n=1,2,3,...$) or the harmonic oscillator ($\infty < x < \infty$ and $n=0,1,2,...$)

orthonormal

$\langle \psi_n | \psi_n \rangle = \int dx \psi_n^*(x) \psi_n(x) = 1$
$\langle \psi_n | \psi_m \rangle = \int dx \psi_n^*(x) \psi_m(x) = 0$ for $m \neq n$

$\psi(x) = \sum_n c_n \psi_n(x)$ \{
completeness
$c_n = \int dx \psi_n^*(x) \psi(x)$

New notation:

$|\psi\rangle = \sum_n c_n |\psi_n\rangle$

$c_n = \langle \psi_n | \psi \rangle$
Just as a 2D vector \( \vec{v} = c_1 \hat{x} + c_2 \hat{y} \) may be specified by its components \( c_1 \) and \( c_2 \), after a basis is chosen, we can specify vectors in the function space by the \( c_n \).

Convention: write as a column vector

\[
|\psi\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}
\]

It is also conventional to have a dual vector written as

\[
<\psi| = (c_1^* \ c_2^* \ c_3^* \ \cdots)
\]

Using matrix multiplication

\[
<\psi|\psi> = (c_1^* \ c_2^* \ \cdots) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} = \int dx \, \psi^*(x) \psi(x)
\]

Dirac bracket notation:

\[
<\psi| \psi>
\]