(1) Consider a hydrogen atom whose unperturbed Hamiltonian,

\[ H^0 = \frac{1}{2m} \left[ p_x^2 + p_y^2 + p_z^2 \right] - \frac{e^2}{4\pi\varepsilon_0 r}, \]

is perturbed by the nearby atoms, the interaction with which can be approximated as,

\[ H' = \beta(3z^2 - r^2). \]

Recall that the unperturbed \( n = 1 \) and \( n = 2 \) hydrogen wave functions are,

\[
\psi_{100} = \frac{1}{\sqrt{\pi}a^3} e^{-r/a}, \quad \psi_{200} = \frac{1}{\sqrt{8\pi}a^3} \left[ 1 - \frac{r}{2a} \right] e^{-r/2a},
\]

\[
\psi_{210} = \frac{1}{\sqrt{32\pi}a^3} \cos(\theta) e^{-r/2a}, \quad \psi_{21\pm 1} = \frac{1}{\sqrt{64\pi}a^3} \sin(\theta) e^{\pm i\phi r/2a},
\]

where \( a \equiv 4\pi\varepsilon_0 h^2/(e^2m) \) is the Bohr radius.

a) What are the dimensions of the constant \( \beta? \) (5 points)
b) What is the first order correction to the ground state energy? (10 points)
c) What are the first order corrections to the \( n = 2 \) state energies? (10 points)

(2) Use a trial wave function of the form,

\[
\psi(x) = \begin{cases} 
B \sin\left(\frac{\pi x}{a}\right), & \text{if } (-a < x < a), \\
0, & \text{otherwise}
\end{cases}
\]

to obtain a bound on the first excited state of a one-dimensional harmonic oscillator. (30 points)

(3) Use the WKB approximation in the form

\[
\int_{r_1}^{r_2} dp(r) = \left( n - \frac{1}{2} \right) \pi \hbar
\]

to estimate the bound state energies for hydrogen. (30 points) Don’t forget the centrifugal term in the effective potential

\[
V_{\text{eff}}(r) = -\frac{e^2}{4\pi\varepsilon_0 r} + \frac{\hbar^2}{2m} \frac{\ell(\ell+1)}{r^2}.
\]

The following integral may help:

\[
\int_{a}^{b} dx \sqrt{\frac{(x-a)(b-x)}{x}} = \frac{\pi}{2} \left( \sqrt{b} - \sqrt{a} \right)^2.
\]