Poisson Variables

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Poisson processes

A *Poisson process* is one in which events are randomly distributed in time, space or some other variable with the number of events in any non-overlapping intervals statistically independent. For example, naturally occurring gamma rays detected in a scintillation detector are randomly distributed in time, or chocolate chips in a cookie dough are randomly distributed in volume. For simplicity, we will limit our discussion to events randomly distributed in time.

A *homogeneous* Poisson process is one in which the long-term average event rate is constant. The average rate will be denoted $\Gamma$ and in any interval $\Delta t$ the expected number of events is

$$\mu = \Gamma \Delta t$$

A *nonhomogeneous* Poisson process is one in which the average rate of events changes and so might be expressed as some function $\Gamma(t)$ of time. The number of events expected in an interval from $t_1$ to $t_2$ would then be the integral

$$\mu = \int_{t_1}^{t_2} \Gamma(t) dt$$

While the expected number of events $\mu$ for a given experiment need not be an integer, the number of events $n$ actually observed must be. Moreover, due to the randomness of the events, $n$ may be more or less than $\mu$ with the probability for a given $n$ depending only on $\mu$ and given by

$$P(n) = e^{-\mu} \frac{\mu^n}{n!}$$

This is the Poisson distribution, a discrete probability distribution, with each $P(n)$ giving the probability for that particular $n$ to occur.

Two Poisson distributions for $\mu = 1.5$ and $\mu = 100$ are shown in Fig. 1. This figure shows the parent distributions. Real sample distributions would be expected to vary somewhat from the parent, getting closer to the parent as the sample size $N$ increases.
The exponential probability density function

A better way of describing $\Gamma$ is as a probability per unit time that an event will occur. That is

$$dP = \Gamma dt$$ (4)

where $dP$ is the differential probability that an event will occur in the infinitesimal time interval $dt$. Of course, some care must be taken when translating a rate to a probability per unit time. For example, if $\Gamma = 10/s$, it is obviously not true that the probability is 10 that an event will occur in any particular second. However, if that same rate is expressed $\Gamma = 0.01/ms$ it is roughly true that the probability is 0.01 that an event will happen in any particular millisecond. Eq. 4 only becomes exact in the limit of infinitesimal $dt$.

Equation 4 also describes the decay process of an excited state of an atom, nuclei, or subatomic particle. In these cases, $dP = \Gamma dt$ is the probability for the excited state to decay in the next time interval $dt$ and $\Gamma$ is called the decay rate for the excited state rather than an event rate.

Equation 4 can be shown to lead directly to the Poisson probability distribution. The first step is to see how it leads to the exponential probability density function (pdf) giving the probability $dP_e(t)$ that the next Poisson event (or the decay of an excited state) will occur in the interval from $t$ to $t + dt$.

If the probability of no event (or survival of the excited state) to a time $t$ is denoted $P(0; t)$, then the probability of no event (or survival) to $t + dt$ would be the product of this probability with the probability of no event (or no decay) in the interval $dt$ following $t$. Since the probability of an event (or decay) in this interval is $\Gamma dt$, the probability of no event (or no decay) in this interval is $1 - \Gamma dt$ and thus:

$$P(0; t + dt) = P(0; t)(1 - \Gamma dt)$$ (5)

Rearranging and substituting $(P(0; t + dt) - P(0; t))/dt = dP(0; t)/dt$ gives

$$\frac{dP(0; t)}{dt} = -\Gamma P(0; t)$$ (6)

\footnote{Equation 4 is equivalent to $dP_e(0) = \Gamma dt$ and must be the $t = 0$ limiting case for the general solution.}
which has the general solution $P(0; t) = Ae^{-\Gamma t}$. Because we must start with no event (or no decay) at $t = 0$, $P(0; 0) = 1$ and so $A = 1$ giving

$$P(0; t) = e^{-\Gamma t} \quad (7)$$

Then, the differential probability $dP_e(t)$ for the next event (or decay) to occur in the interval from $t$ to $t + dt$ is given by the probability of no event (or no decay) in the interval from 0 to $t$ followed by an event (or a decay) in the next interval $dt$. The former has a probability $P(0; t) = e^{-\Gamma t}$ and the later has a probability $\Gamma dt$. Thus

$$dP_e(t) = \Gamma e^{-\Gamma t} dt \quad (8)$$

Equations 8 is a continuous probability density function (pdf). It is properly normalized, i.e., the integral over all times from 0 to $\infty$ is unity as required. It also has the very reasonable property that the expectation value for the random variable $t$—the time to the next event (or to the decay)—is given by

$$\langle t \rangle = \int_0^\infty t \Gamma e^{-\Gamma t} dt = \frac{1}{\Gamma} \quad (9)$$

In the case of decay, the expectation value $\langle t \rangle$, henceforth denoted $\tau$, is called the *lifetime* of the excited state. Thus, $\Gamma$ and $\tau$ are equivalent ways to quantify the decay process. If the decay rate is 1000/s, the lifetime is 0.001 s. Moreover, Eq. 8 is often expressed in terms of the lifetime rather than the decay rate.

$$dP_e(t) = \frac{1}{\tau} e^{-t/\tau} dt \quad (10)$$

The probability for decay in a time $\tau$ is found by integrating Eq. 8 from 0 to $\tau$ and gives the value $1/e$. Thus, for a large sample of excited states at $t = 0$, the fraction $1/e$ of them will have decayed by $\tau$. The time it would take for half the sample to decay is called the half-life and is easily shown to be $\tau \ln 2$.

**The Poisson probability distribution**

There are several possible derivations of the Poisson probability distribution. It is often derived as a limiting case of the binomial probability distribution. The derivation to follow relies on Eq. 4 and begins with Eq. 7 for the probability $P(0; t)$ that there will be no events in some finite interval $t$.

Next, a recursion relation is derived for the probability, denoted $P(n + 1; t)$, for there to be $n + 1$ events in a time $t$, which will be based on the probability $P(n; t)$ of one less event. For there to be $n + 1$ events in $t$, three independent events must happen in the following order (their probabilities given in parentheses).
• There must be \( n \) events up to some point \( t' \) in the interval from 0 to \( t \) \((P(n,t')\) by definition).

• An event must occur in the infinitesimal interval from \( t' \) to \( t' + dt' \) \((\Gamma dt'\) by Eq. 4).

• There must be no events in the interval from \( t' \) to \( t \) \((P(0,t-t')\) by definition).

The probability of \( n + 1 \) events in the interval from 0 to \( t \) would be the product of the three probabilities above integrated over all \( t' \) from 0 to \( t \) to take into account that the last event may occur at any time in the interval. That is,

\[
P(n + 1; t) = \int_0^t P(n; t')\Gamma dt' P(0; t-t')
\]  \hspace{1cm} (11)

From Eq. 7 we already have \( P(0; t-t') = e^{-\Gamma(t-t')} \) and thus Eq. 11 becomes

\[
P(n + 1; t) = \Gamma \int_0^t P(n; t')e^{-\Gamma(t-t')}dt'
\]  \hspace{1cm} (12)

With the following definition:

\[
P(n; t) = e^{-\Gamma t}P(n; t)
\]  \hspace{1cm} (13)

Eq. 12 becomes

\[
e^{-\Gamma t}P(n + 1; t) = \Gamma \int_0^t e^{-\Gamma t'}P(n; t')e^{-\Gamma(t-t')}dt'
\]

\[
P(n + 1; t) = \Gamma \int_0^t P(n; t')dt'
\]  \hspace{1cm} (14)

From Eqs. 7 and 13, \( P(0; t) = 1 \) and then \( P(1; t) \) can be found from an application of Eq. 14

\[
P(1, t) = \Gamma \int_0^t P(0, t')dt'
\]

\[
= \Gamma \int_0^t dt'
\]

\[
= \Gamma t
\]  \hspace{1cm} (15)

Applying Eq. 14 for the next few terms

\[
P(2, t) = \Gamma \int_0^t P(1; t')dt'
\]

\[
= \Gamma \int_0^t \Gamma t'dt'
\]

\[
= \frac{\Gamma^2 t^2}{2}
\]  \hspace{1cm} (16)
\[ P(3, t) = \Gamma \int_0^t P(2, t') dt' \]
\[ = \Gamma \int_0^t \frac{\Gamma^2 t^2}{2} dt' \]
\[ = \frac{\Gamma^3 t^3}{2 \cdot 3} \]

(17)

The pattern clearly emerges that

\[ P(n; t) = \frac{(\Gamma t)^n}{n!} \]

(18)

And thus with Eq. 13, the Poisson probabilities become

\[ P(n; t) = e^{-\Gamma t} \frac{(\Gamma t)^n}{n!} \]

(19)

Note that, as expected, the right side depends only on the combination \( \mu = \Gamma t \). Using that substitution and dropping the now meaningless \( t \) dependence on the left, the standard Poisson probability distribution of Eq. 3 results.

Although the Poisson probabilities were derived assuming a homogeneous process, they are also appropriate for nonhomogeneous processes with the appropriate value of \( \mu \) (Eq. 2).

Several important properties of the Poisson distribution are easily investigated. For example, the normalization condition—that some value of \( n \) from 0 to infinity must occur—translates to

\[ \sum_{n=0}^{\infty} P(n) = 1 \]

(20)

and is easily verified from the series expansion of the exponential function

\[ e^\mu = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \]

(21)

The expected number of events is found from the following weighted average

\[ \langle n \rangle = \sum_{n=0}^{\infty} n P(n) \]

(22)

and is evaluated as follows:

\[ \langle n \rangle = \sum_{n=1}^{\infty} n e^{-\mu} \frac{\mu^n}{n!} \]
\[ = \mu \sum_{n=1}^{\infty} e^{-\mu} \frac{\mu^{n-1}}{(n-1)!} \]
\[ = \mu \sum_{m=0}^{\infty} e^{-\mu} \frac{\mu^m}{m!} \]
\[ = \mu \]

(23)
In the first line, the explicit form for $P(n)$ is used and the first term $n = 0$ is explicitly dropped as it does not contribute to the sum. In the second line, the numerator’s $n$ is canceled with the one in the denominator’s $n!$ and one $\mu$ is also factored out in front of the sum. In the third line, $m = n - 1$ is substituted, forcing a change in the indexing from $n = 1...\infty$ to $m = 0...\infty$. And in the last line, the normalization property is used.

Thus, the Poisson probability distribution gives the required result that the expectation value (or parent average) of $n$ is equal to $\mu$.

We should also want to investigate the standard deviation of the Poisson distribution which would be evaluated from the expectation value

$$\sigma^2 = \langle (n - \mu)^2 \rangle = \langle n^2 \rangle - \mu^2$$

(24)

The expectation value $\langle n^2 \rangle$ is evaluated as follows:

$$\langle n^2 \rangle = \sum_{n=0}^{\infty} n^2 P(n)$$

$$= \sum_{n=1}^{\infty} n^2 e^{-\mu} \frac{\mu^n}{n!}$$

$$= \mu \sum_{n=1}^{\infty} ne^{-\mu} \frac{\mu^{n-1}}{(n-1)!}$$

$$= \mu \sum_{m=0}^{\infty} (m+1)e^{-\mu} \frac{\mu^m}{m!}$$

$$= \mu \left[ \sum_{m=0}^{\infty} me^{-\mu} \frac{\mu^m}{m!} + \sum_{m=0}^{\infty} e^{-\mu} \frac{\mu^n}{n!} \right]$$

$$= \mu [\mu + 1]$$

$$= \mu^2 + \mu$$

(25)

In the second line, the form for $P(n)$ is substituted and the first term $n = 0$ is dropped from the sum as it does not contribute. In the third line, one power of $\mu$ is factored out of the sum and one $n$ is canceled against one in the $n!$. The indexing and lower limit of the sum is adjusted in the fourth line using $m = n - 1$. In the fifth line, the $m + 1$ term is separated into two terms, which are evaluated separately in the sixth line; the first term is just $\langle m \rangle$ and evaluates to $\mu$ by Eq. 23 and the second term evaluates to 1 based on the normalization condition—Eq. 20.

Now combining Eq. 25 with Eq. 24 gives the result

$$\sigma^2 = \mu$$

(26)

implying that the parent variance is equal to the mean, i.e., that the standard deviation is given by $\sigma = \sqrt{\mu}$. 