Conditions under which the stress tensor for a point particle is conserved

We take the energy-momentum of the point particle to be given by:

\[ T^{\mu\nu}(x) = m \int \frac{\delta^4(x^\alpha - z^\alpha(\tau))}{\sqrt{-g(z)}} \dot{z}^\mu \dot{z}^\nu d\tau. \]

Then

\[
\nabla_\mu T^{\mu\nu}(x) = \frac{\partial T^{\mu\nu}(x)}{\partial x^\mu} + \Gamma^\mu_{\mu\sigma}(x) T^{\sigma\nu}(x) + \Gamma^\nu_{\mu\sigma}(x) T^{\mu\sigma}(x)
\]

\[
= m \int d\tau \left[ \frac{\partial \delta^4(x^\alpha - z^\alpha(\tau))}{\partial x^\mu} \frac{\dot{z}^\mu \dot{z}^\nu}{\sqrt{-g(z)}} + \left( \Gamma^\mu_{\mu\sigma}(x) \dot{z}^\nu + \Gamma^\nu_{\mu\sigma}(x) \dot{z}^\mu \right) \frac{\delta^4(x^\alpha - z^\alpha(\tau))}{\sqrt{-g(z)}} \right]
\]

\[
= m \int d\tau \left[ - \frac{d\delta^4(x^\alpha - z^\alpha(\tau))}{d\tau} \frac{\dot{z}^\nu}{\sqrt{-g(z)}} + \left( \Gamma^\mu_{\mu\sigma}(x) \dot{z}^\nu + \Gamma^\nu_{\mu\sigma}(x) \dot{z}^\mu \right) \frac{\delta^4(x^\alpha - z^\alpha(\tau))}{\sqrt{-g(z)}} \right]
\]

\[
= m \int d\tau \delta^4(x^\alpha - z^\alpha(\tau)) \left[ \frac{d}{d\tau} \left( \frac{\dot{z}^\nu}{\sqrt{-g(z)}} \right) + \left( \Gamma^\mu_{\mu\sigma}(x) \dot{z}^\nu + \Gamma^\nu_{\mu\sigma}(x) \dot{z}^\mu \right) \frac{\dot{z}^\sigma}{\sqrt{-g(z)}} \right]
\]

\[
= m \int d\tau \dot{z}^\sigma \left[ \nabla_\sigma \dot{z}^\nu + \left( \Gamma^\mu_{\mu\sigma}(x) - \Gamma^\nu_{\mu\sigma}(z) \right) \dot{z}^\mu \right] \frac{\delta^4(x^\alpha - z^\alpha(\tau))}{\sqrt{-g(z)}}
\]

The last two terms in square brackets are zero because the \( \delta \)-function multiplies terms which vanish at coincidence — for any orbit \( z^\alpha(\tau) \). Thus, if \( z(\tau) \) is a geodesic, \( \nabla_\mu T^{\mu\nu}(x) = 0 \).

Note:

- The second equality holds because the derivative acts only on the \( \delta \)-function.
- The third equality holds by introducing a change of sign upon swapping the arguments of the derivative on the \( \delta \)-function and using \( \dot{z}^\mu \partial / \partial x^\mu = d/d\tau \).
- The fourth equality holds from performing integration by parts.
- The fifth equality holds by rearrangement of terms after completing the covariant derivative on \( \dot{z}^\nu \) and using \( d\ln \sqrt{-g(z)} / d\tau = \Gamma^\mu_{\mu\sigma}(z) \dot{z}^\sigma \).
- The result given holds, and somewhat simpler, even if, from the beginning, \( g(z) \Rightarrow g(x) \).