I. BEHAVIOUR AND SOLUTIONS OF
ORDINARY DIFFERENTIAL EQUATIONS

Suppose we have a general second order operator

\[
\left( \frac{d^2}{dx^2} + p \frac{d}{dx} + q \right) y = 0 .
\]  

(1.1)

Let us substitute the following

\[
y = e^\int f(x) dx .
\]  

(1.2)

One finds (prime denoting \( \frac{d}{dx} \))

\[
\frac{y'}{y} = f + \frac{z'}{z}
\]  

(1.3)

\[
\frac{y^{(n+1)}}{y} = \left( \frac{y^{(n)}}{y} \right)' + \frac{y^{(n)} y'}{y y'}
\]  

(1.4)

Thus, we get

\[
\frac{y''}{y} = f' + \frac{z''}{z} + f^2 + 2 f \frac{z'}{z}
\]  

(1.5)

and the main result

\[
z'' + (p + 2f)z' + (q + f' + pf + f^2)z = 0 .
\]  

(1.6)

Now let us put in the condition

\[
f = -\frac{1}{2}p
\]  

(1.7)

so that

\[
z'' + \left( q - \frac{1}{2}p' - \frac{1}{4}p^2 \right) z = 0
\]  

(1.8)

and we get the invariant

\[
Q = q - \frac{1}{2}p' - \frac{1}{4}p^2
\]  

(1.9)

of the equivalence class of equations.

Let us do the following change, \( x \rightarrow r(x) \), such that \( z'' + Qz = 0 \) becomes ( \( \dot{\cdot} \) denotes \( \frac{d}{dr} \))

\[
(r')^2 \ddot{z} + r'' \dot{z} + Qz = 0 .
\]  

(1.10)
The domains over which solutions can be written may be quite different; i.e. \((-\pi, \pi) \rightarrow (-\infty, \infty)\). Now, Eq. (1.10) can be rewritten as

\[
(r')^2 \left( \ddot{z} + \frac{r''}{(r')^2} \dot{z} + \frac{Q}{(r')^2} z \right) = 0.
\]  

(1.11)

Noting that

\[
\frac{d}{dr} = \frac{1}{r} \frac{d}{dx}
\]

we can write

\[
-\frac{1}{2r} \left( \frac{r'''}{r'} - \frac{2}{3} \left( \frac{r''}{r'} \right)^2 \right) - \frac{1}{2r} \left( \frac{r'''}{r'} - \frac{3}{2} \left( \frac{r''}{r'} \right)^2 \right) = \frac{1}{(r')^2} \frac{d}{dx} \left( \frac{r'}{r'} - \frac{1}{2} \right).
\]  

(1.13)

Using Eq. (1.13) and the transformation \(z \rightarrow ze^{\frac{1}{2} \int \frac{r''}{r'} dx}\) we get

\[
\left( \frac{d^2}{dr^2} + \frac{1}{(r')^2} \left( Q + \frac{\left( \frac{r'}{r'} - \frac{1}{2} \right)''}{(r')^2} \right) \right) ze^{\frac{1}{2} \int \frac{r''}{r'} dx} = 0.
\]  

(1.14)

(The Schwarzian derivative is given by

\[
\{r, x\} \equiv \left( \frac{r'''}{r'} - \frac{3}{2} \left( \frac{r''}{r'} \right)^2 \right),
\]  

(1.15a)

\[
\{x, r\} = \frac{1}{(r')^2} \{r, x\}.
\]  

(1.15b)

For some function \(S\)

\[
\{S, x\} \equiv \{S, r\}(r')^2 + \{r, x\}.
\]  

(1.16)

Note that

\[
e^{\frac{1}{2} \int \frac{r''}{r'} dx} = e^{\left( \ln \frac{r''}{r'} \right)^{\frac{1}{2}}} = \sqrt{r'}.
\]  

(1.17)

Thus Eq. (1.14) becomes

\[
\left( \frac{d^2}{dr^2} + \frac{1}{(r')^2} \left( Q + \frac{\left( \frac{r'}{r'} - \frac{1}{2} \right)''}{(r')^2} \right) \right) \sqrt{r'} z = 0.
\]  

(1.18)

Suppose that \(r\) is chosen such that the second term of Eq. (1.18) is zero, then a solution \(z\) of Eq. (1.18) is

\[
z = (r')^{-\frac{1}{2}}.
\]  

(1.19)
A linearly independent solution of \( \frac{d^2}{dr^2} \sqrt{r} z = 0 \) is
\[
z = r (r')^{-\frac{1}{2}}. \tag{1.20}
\]

Now suppose that the second term of Eq. (1.18) is equal to one, then we find for
\[
\left( \frac{d}{dr^2} + 1 \right) \sqrt{r} z = 0
\]
that
\[
z = e^{\pm ir} \sqrt{r'} \tag{1.21}
\]

Eq. (1.21) is similar in form to a solution like
\[
G = \frac{e^{\pm \int \sqrt{F} dx}}{F^{\frac{1}{4}}}. \tag{1.22}
\]

One uses this in a W.K.B. approximation. Every solution can be written in the form Eq. (1.22). As an example we can write
\[
\frac{\sin r}{\sqrt{r'}} = \frac{e^{ig}}{\sqrt{g'}}. \tag{1.23}
\]

This \( g \) must be a non-trivial complex function.
II. FORM OF SOLUTIONS NEAR REGULAR SINGULAR POINTS

For
\[ z'' + Qz = 0 , \]  
(2.1)
if we have a solution \( z_1 \) we can always find another solution
\[ z_2 = z_1 \int \frac{1}{(z_1)^2} dx . \]  
(2.2)

For
\[ y'' + py' + qy = 0 , \]  
(2.3)
we have the analogous second solution
\[ y_2 = y_1 \int \frac{W(x)}{(y_1)^2} dx . \]  
(2.4)

Consider the following (form of the confluent hypergeometric) equation
\[ z'' + \left(-\alpha^2 + \frac{2\alpha\beta}{x} + \frac{1}{x^2} - \gamma^2 \right)z = 0 . \]  
(2.5)
This has a regular singular point at \( x = 0 \) and an irregular singular point at \( x = \infty \). Now, for \( x \to 0 \), \( z \) will have the behaviour
\[ z \to x^\frac{1}{2} + \epsilon' \gamma, \quad \epsilon' = \pm 1 \]  
(2.6a)
and for \( x \to \infty \), \( z \) will have the behaviour
\[ z \to e^{\epsilon \alpha x}x^{-\epsilon \beta}, \quad \epsilon = \pm 1 . \]  
(2.6b)

For an arbitrary point (ordinary) we can impose whatever boundary conditions we want
\[ x \to x_0 : z = 1, \quad (x - x_0) . \]

For the hypergeometric equation we would consider
\[ x(x - 1)z'' + \left(\frac{1}{4} - \alpha^2 \right) - \frac{1}{x} - \frac{1}{x - 1} + \frac{1}{x - 1} \right) z = 0. \]  
(2.7)
The behaviour near \( x \to 1, 0, \infty \) will be
\[ x \to 1 : (x - 1)^{\frac{1}{2} + \epsilon'' \gamma}, \quad \epsilon'' = \pm 1 . \]  
(2.8a)
\[ x \rightarrow 0 : x^{1/2 + \epsilon' \beta}, \epsilon' = \pm 1 \]
\[ x \rightarrow \infty : x^{1/2 + \epsilon \alpha}, \epsilon = \pm 1. \]

Now let us look at solving equations like Eq. (2.5). Suppose we factor out the quantity
\[ z = x^{1/2 + \epsilon' \gamma} e^{\epsilon \alpha x} u(x) \]
where \( u(x) \) is some series. We will find that if
\[ \frac{1}{2} + \epsilon' \gamma + N \equiv -\epsilon \beta \]
then the series will truncate. It is most important (necessary) for truncation that \( N \) be an integer. Putting Eq. (2.9) in Eq. (2.5) we get
\[ u'' + 2\left( \epsilon \alpha + \frac{1}{2} + \epsilon' \gamma \right) u' + \frac{2\epsilon \alpha (\epsilon \beta + \frac{1}{2} + \epsilon' \gamma)}{x} u = 0. \]

Now let
\[ u = \sum_n a_n x^n \]
so that we get the following recurrence relation
\[ (n + 1) na_{n+1} + 2\epsilon \alpha (\frac{1}{2} + \epsilon \beta + \epsilon' \gamma) a_n + (\frac{1}{2} + \epsilon' \gamma) (n + 1) a_{n+1} + 2\epsilon \alpha n a_n = 0. \]

Truncation occurs if there exists an \( N \) such that
\[ a_{n+1} \equiv 0 \]
which is implied by Eq. (2.10).

Let us operate on \( x^*(\text{Eq. (2.11)}) \) \( n \) times with \( \frac{d}{dx} \). We will get
\[ \left( \left( x \frac{d^2}{dx^2} + 2\left( \epsilon \alpha x + \frac{1}{2} + \epsilon' \gamma + \frac{1}{2} n \right) \frac{d}{dx} + 2\epsilon \alpha \left( \epsilon \beta + \frac{1}{2} + \epsilon' \gamma \right) + 2\epsilon \alpha n \right) \frac{d^n}{dx^n} \right) u = 0 \]
Note that this equation is always second order in the \( n^{th} \) derivative, and that when \( \epsilon \beta + \frac{1}{2} + \epsilon' \gamma \) is a negative integer \(-N\), then one solution for \( \frac{d^N u}{dx^N} \) is a constant (i.e., gives the truncating solution). Also, from Eq. (2.15) we see that \( \frac{d^n}{dx^n} \) acting on \( u \) is a shift (cf. raising or lowering) operator, with the action
\[ \gamma \rightarrow \epsilon' \gamma + \frac{\epsilon' n}{2}, \beta \rightarrow \epsilon' \beta + \frac{\epsilon n}{2}. \]
In Quantum Mechanics, Eq. (2.7) arises for the (spin-weighted) spherical harmonics with
\[
\alpha = \ell + \frac{1}{2} \\
\beta = \frac{1}{2}(s - m) \\
\gamma = \frac{1}{2}(s + m),
\]
and Eq. (2.5) arises for the Coulomb (radial) wave functions, with bound state occurrence being given by a condition for truncating solutions.

From Eq. (2.7) let us now pull out the factor
\[
z = x^{\frac{1}{2} + \epsilon' \beta} (x - 1)^{\frac{1}{2} + \epsilon'' \gamma} u(x)
\]
then (after several lines of algebra) we will find the condition for truncation is:
\[
\frac{1}{2} + \epsilon' \beta + \frac{1}{2} + \epsilon'' \gamma + N = \frac{1}{2} + \epsilon \alpha.
\]
In general, we find that \( \frac{d^k u}{dx^k} \) gives rise to a solution of Eq. (2.5) with \( \beta \rightarrow \beta + \epsilon' \frac{k}{2}; \gamma \rightarrow \gamma + \epsilon'' \frac{k}{2} \).

Raising and lowering operator characteristics are determined entirely by techniques considered above.
III. RECURRENCE RELATIONS

In general, a recurrence relation has the form

\[ \ddot{y} = \alpha(x)y + \beta(x) \frac{\partial y}{\partial x}. \]  (3.1)

We want to devise a method, in principle, for understanding why recurrence relations exist and what characterizes their behaviour.

Rewrite Eq. (3.1) as

\[ \ddot{y} = \beta \left( \frac{\partial}{\partial x} + \frac{\alpha}{\beta} \right) y, \]  (3.2)

or alternatively

\[ \ddot{y} = B \frac{\partial}{\partial x} Ay. \]  (3.3)

In the simplest recurrence relation for Eq. (2.5) above we would have

\[ \frac{\alpha}{\beta} \sim \mu + \frac{\lambda}{x}, \]  (3.4a)

and for Eq. (2.7)

\[ \frac{\alpha}{\beta} \sim \frac{\sigma}{x} + \frac{\tau}{x - 1}. \]  (3.4b)

Then we would have

\[ A(x) = e^{\mu x} x^\lambda \text{ or } x^\sigma (x - 1)^\tau \]  (3.5)

as given by the constructions indicated in the previous section.

Consider the two equations

\[ (\partial_{xx} + q(x))y = 0 \]  (3.6a)
\[ (\partial_{xx} + Q(x))z = 0. \]  (3.6b)

We want to seek \( \alpha, \beta \) (which truncate) such that

\[ y = \alpha z + \beta z'. \]  (3.7)

Let Eq. (3.6a–b) be anything we wish to write down. Suppose that two linearly independent solutions \( y_1, y_2 \) exist for Eq. (3.6a). Also, let the same be true for \( z_1, z_2 \) of Eq. (3.6b). A possible, but not useful, situation is when Eq. (3.7) would map \( z_1 \rightarrow y_1 \) but would permit \( z_2 \rightarrow \) anything, e.g.,

\[ y_1 = \left( f(x) \left( \partial_x - \frac{z'_1}{z_1} \right) \right) + \frac{y_1}{z_1} z_1. \]  (3.8)
However, \( f(x) \) has no restrictions so that Eq. (3.8) is not very useful – it is too general.

Consider the equations
\[
y_1 = \alpha z_1 + \beta z_1' \\
y_2 = \alpha z_2 + \beta z_2'.
\]

The solutions for \( \alpha \) and \( \beta \) are
\[
\alpha = \frac{-z_1'y_2 - z_2'y_1}{W(z_1, z_2)} \\
\beta = \frac{z_1y_2 - z_2y_1}{W(z_1, z_2)}.
\]

where \( W(,\) \) is the Wronskian. In general there exists an inverse map
\[
z = \frac{1}{k}(\alpha + \beta')y - \beta y'
\]

where
\[
k = \frac{W(y_1, y_2)}{W(z_1, z_2)} = \text{constant}.
\]

Thus our mapping and its inverse exists and is unique for the Eqs. (3.6a,b). Now
\[
(\partial_{xx} + q(x)) = \left(\partial_x + \frac{\alpha + \beta'}{\beta}\right) \left(\partial_x - \frac{\alpha + \beta'}{\beta}\right) + \frac{k}{\beta^2}
\]
\[
(\partial_{xx} + Q(x)) = \left(\partial_x - \frac{\alpha}{\beta}\right) \left(\partial_x + \frac{\alpha}{\beta}\right) + \frac{k}{\beta^2}.
\]

Thus
\[
q = \frac{k}{\beta^2} - \left(\frac{\alpha + \beta'}{\beta}\right) - \left(\frac{\alpha + \beta'}{\beta}\right)^2
\]
\[
Q = \frac{k}{\beta^2} + \left(\frac{\alpha}{\beta}\right)' - \left(\frac{\alpha}{\beta}\right)^2.
\]

When \( \beta \) is a constant we get the simplest examples of “raising” and “lowering” type of operators.

If we choose our mapping as
\[
z_1 \to ay_1 + by_2 \\
z_2 \to cy_1 + dy_2
\]
such that \( ad - bc \neq 0 \), then for \( b = c = 1, a = d = 0, \)
\[
\beta = \frac{y_1z_1 - y_2z_2}{W}.
\]
We find that there are four linearly independent \( \beta \)'s. It is not always clear, however, which \( \beta \) to pick, or that any useful choice is available.

Look at
\[
\beta^2 (\partial_{xx} + q) = (\beta \partial_x + \alpha)(\beta \partial_x - \alpha - \beta') + k \tag{3.17a}
\]
\[
\beta^2 (\partial_{xx} + Q) = (\beta \partial_x - \alpha - \beta')(\beta \partial_x + \alpha) + k \tag{3.17b}
\]
where for Eqs. (3.17a,b) define
\[
A = (\beta \partial_x + \alpha) \tag{3.18a}
\]
\[
B = (\beta \partial_x - \alpha - \beta') \tag{3.18b}
\]
so that
\[
[A, B] \neq 0 . \tag{3.19}
\]

Notice that
\[
B(AB + k) = (BA + k)B \tag{3.20a}
\]
\[
A(BA + k) = (AB + k)B . \tag{3.20b}
\]

Note that, for the operators multiplied by \( \beta^2 \), \( k \) simply represents a shift in the eigenvalues (spectrum) of the operators \( AB \) and \( BA \): this \( \beta^2 \) has really changed us to quite new operators and knowing the zeros of \( \beta \) is very important.
For simplicity, consider a case of the whole interval \((-\infty, \infty)\). Suppose that the behaviour of \(y\) in Eq. (3.6a) is

\[
y_1 = \lim_{x \to -\infty} e^{i\omega x} = \lim_{x \to -\infty} (Ae^{i\omega x} + Be^{-i\omega x}) \tag{0.1a}
\]

\[
y_2 = \lim_{x \to -\infty} e^{-i\omega x} = \lim_{x \to -\infty} (Ce^{i\omega x} + De^{-i\omega x}) \tag{0.1b}
\]

and similarly for \(z\) in Eq. (3.6b). Then \(\beta\) will have the form as \(x \to -\infty\)

\[
\beta = ay_1z_1 + by_1z_2 + cy_2z_1 + dy_2y_2 \\
\simeq e^{-2i\omega x} + \text{const} + \text{const} + e^{2i\omega x}. \tag{0.2}
\]

To avoid a \(\beta\) which oscillates, one would choose \(a = d = 0\). Now, as \(x \to \infty\), \(\beta\) will have the form

\[
\beta = b(Ae^{i\omega x} + Be^{-i\omega x})(Ce^{i\omega x} + De^{-i\omega x}) + c(Ce^{i\omega x} + De^{-i\omega x})(\tilde{A}e^{i\omega x} + \tilde{B}e^{-i\omega x}) \tag{0.3}
\]

where the tildes are associated with solutions \(z_1, z_2\) of \(z\) for Eq. (3.6b). Similarly, we require, at \(\pm\infty\),

\[
bA\tilde{C} + cC\tilde{A} = 0, \tag{0.3a}
\]

\[
bB\tilde{D} + cD\tilde{B} = 0. \tag{0.3b}
\]

This keeps our solutions from oscillating at \(x \to \pm\infty\). Therefore

\[
\frac{b}{c} = -\frac{C\tilde{A}}{A\tilde{C}} = -\frac{D\tilde{B}}{B\tilde{D}} \tag{0.4}
\]

and

\[
\frac{CB}{AD} = \frac{\tilde{C}\tilde{B}}{\tilde{A}\tilde{D}}. \tag{0.5}
\]

Looking at

\[
1 - \frac{CB}{AD} = \frac{1}{AD} = \frac{1}{\tilde{A}\tilde{D}} \tag{0.6}
\]

where, in this case, \(k = 1\). This implies

\[
AD = \tilde{A}\tilde{D} \tag{0.7}
\]

\[
BC = \tilde{B}\tilde{C}. \tag{0.8}
\]
These conditions help to imply that $A, \tilde{A}$ must have the same zeros (poles) and $B, \tilde{B}$ have the same zeros (poles); i.e., that the singular points of the scattering data correspond. Otherwise, $b$ or $c$ would be zero and this would mean that $z_1, z_2$ would map to the same function $y$. Thus, for $\beta$ to be non-oscillatory (it is useless otherwise) we must have, at least, that the singular points of the scattering data correspond. This is in fact a very restrictive condition on $q, Q$ in Eq. (3.6a, 3.6b), so that useful recurrence relations can occur only in special situations. However, precisely these situations can arise in the use of inverse scattering techniques to solve non-linear (completely integrable) evolution equations, which can give multi-parameter potentials (i.e., $q, Q$) for linear equations with trivially related scattering data.

Finally, recall Eq. (3.13b)

$$(\partial_{xx} + Q(x)) = \left( \partial_x - \frac{\alpha}{\beta} \right) \left( \partial_x + \frac{\alpha}{\beta} \right) + \frac{k}{\beta^2}.$$ 

When $k = 0$ we know we have a solution. Once we have this solution we can get all the solutions for $y$ and $z$. Similarly, once we have only two solutions $\beta$, we can solve for everything (even though $\beta$ can be shown to satisfy a fourth order differential equation!).