Due at the start of class on Monday, October 19.

Answer all questions. Please write neatly and include your name on the front page of your answers. To gain maximum credit you should explain your reasoning and show all working.

1. A bead of mass $m$ is constrained to move on a hoop of radius $R$. The hoop rotates with constant angular velocity $\omega$ around a diameter of the hoop, which is a vertical axis (line along which gravity acts).

   a) Set up the Lagrangian and obtain the equations of motion for the bead.

   b) Find the critical angular velocity $\Omega$ below which the bottom of the hoop provides a stable equilibrium for the bead.

   c) Find the stable equilibrium position for $\omega > \Omega$.

2. Two carts are attached to fixed walls by the springs labeled $k_1$ and $k_3$ and to each other by $k_2$. The carts positions $x_1$ and $x_2$ are measured from their respective equilibrium positions.

   a) Write down the Lagrangian for the system and show that the equations of motion can be written in the matrix form $T\ddot{x} = -Vx$, where $x$ is the 2 x 1 column made up of $x_1$ and $x_2$.

   b) Letting $m_1 = m_2 = m$, $k_1 = k_3 = k$ and $k_2 = 3k$, solve the eigenvalue problem and show that $\omega^2$ for this problem is always real, and always positive.

   c) Find the eigenvectors and normal modes of the system.

3. A massless spring (force constant $k_1$) is suspended from the ceiling, with a mass $m_1$ hanging from its lower end. A second spring (force constant $k_2$) is suspended from $m_1$, and a second mass $m_2$ is suspended from the second spring’s lower end. Assume that the masses move only in the vertical direction.

   a) Use coordinates $y_1$ and $y_2$ measured from the masses’ equilibrium positions (and ignoring gravity), write down the Lagrangian for the system.

   b) Show that the equations of motion can be written in the matrix form $T\ddot{y} = -Vy$, where $y$ is the 2 x 1 column made up of $y_1$ and $y_2$.

   c) By a suitable substitution, solve the eigenvalue problem and show that $\omega^2$ for this problem is always real, and always positive.
d) By including gravity, AND using the full extension of the springs from their un-
extended lengths, show that your Lagrangian in a) is justified.

4. As a model of a linear triatomic molecule (such as CO$_2$), consider a system with two identical atoms of mass $m$ connected by two identical springs to a single atom of mass $M$. To simplify matters, assume that the system is confined to move in one dimension.

   a) Write down the Lagrangian and find the normal frequencies of the system. Show that one of the normal frequencies is zero.

   b) Find and describe the motion in the normal modes whose frequencies are nonzero.

   c) Find and describe the motion in the mode with zero frequency.

5. A plane triatomic molecule consists of equal masses $m$ at the vertices of an equilateral triangle with sides of length $a$. Assume that the molecule is held together by spring-like forces between each pair of masses and that the effective spring constants are all equal to $k$. Consider motion only in the plane of the molecule and work in a frame where the center of mass remains fixed.

   a) Without writing down any dynamical equations, find the number of normal modes and explain the reason for your answer. How many of the eigenfrequencies are zero.

   b) One of the normal modes corresponds to a symmetrical stretching of all three edges of the molecule. Find the frequency of this mode.

   c) How many distinct eigenfrequencies do you think the system has? Use an argument based on symmetry to find another non-zero eigenfrequency.