Exam 3 Solutions

1. For a certain driven series RLC circuit, the maximum generator EMF is 125V and the maximum current is 3.20A. The current leads the generator EMF by 60º.

(a) [6 points] What is the impedance of the circuit?

\[ i_m = \frac{\varepsilon_m}{Z} \Rightarrow Z = \frac{\varepsilon_m}{i_m} = \frac{125V}{3.2A} = 39.1\Omega \]

(b) [6 points] What is the resistance of the circuit?

*If the current leads the EMF, it means \( \phi \) is negative:*

\[ \varepsilon(t) = \varepsilon_m \sin \omega t \]

\[ i(t) = i_m \sin(\omega t - \phi) \]

\[ \phi = -60^\circ \]

\[ \tan \phi = \frac{X_L - X_C}{R} \Rightarrow R \tan \phi = X_L - X_C \]

\[ Z = \sqrt{R^2 + (X_L - X_C)^2} \]

\[ Z^2 = R^2 + R^2 \tan^2 \phi \]

\[ R = \frac{Z}{\sqrt{1 + \tan^2 \phi}} = 19.5\Omega \]

(c) [4 points] Is the circuit predominantly capacitive or inductive, and why?

*Since \( \tan \phi < 0 \Rightarrow X_L < X_C \), the circuit is capacitive.*
2. [8 points] An alternating source drives a series RLC circuit with an EMF of maximum amplitude 6.0V. The phase angle of the current is +30º. When the potential difference across the capacitor reaches its maximum positive value of +5V, what is the potential difference across the inductor (including sign)?

\[ \varepsilon(t) = \varepsilon_m \sin \omega t \]
\[ i(t) = i_m \sin (\omega t - \phi) \]
\[ \phi = 30^\circ \]

Since the potential difference across the capacitor is a maximum, that only occurs when the current is at zero (and about to switch directions to drain the capacitor).

\[ \Rightarrow \sin (\omega t - \phi) = 0 \]
\[ \Rightarrow \omega t - \phi = 0, \pi, 2\pi, \ldots \]

We will choose \( \pi \) as follows. The potential across the capacitor is:

\[ V_c = \frac{q}{C} = \int \frac{i}{C} = -\frac{1}{\omega C} \cos (\omega t - \phi) \]

Since by trigonometry \( \cos \alpha = -\sin \left( \alpha - \frac{\pi}{2} \right) \), then we can rewrite as:

\[ V_c = \frac{1}{\omega C} \sin \left( \omega t - \phi - \frac{\pi}{2} \right) \]

So for it to be a maximum, we need:

\[ \omega t - \phi - \frac{\pi}{2} = \frac{\pi}{2} \]
\[ \Rightarrow \omega t = \phi + \pi \]

Or any multiple of \( 2\pi \)

Thus,

\[ \varepsilon(t) = \varepsilon_m \sin \omega t = 6 \sin (\phi + \pi) = 6 \sin (210^\circ) = -3 \]

So by Kirchoff’s loop rule (\( V_R = 0 \) since \( i = 0 \)):

\[ \varepsilon - V_R - V_L - V_C = 0 \]
\[ -3 - 0 - V_L - 5 = 0 \]
\[ \Rightarrow V_L = -8V \]
3. [8 points] In a certain region of space there are no magnetic fields present. An electric field does exist, however. If the y-component of the electric field is \( E_y = E_m x \), where \( E_m \) is a constant, what is the x-component of the electric field?

\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0
\]

\[
\nabla \times \vec{F} \equiv \text{curl}(\vec{F}) = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} - \left( \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) \hat{y} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z}
\]

\[
\Rightarrow \nabla \times \vec{E} = \left( \frac{\partial E_y}{\partial y} - \frac{\partial E_x}{\partial z} \right) \hat{x} - \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{y} = (E_m - E_m) = 0
\]

\[
\Rightarrow \frac{\partial E_x}{\partial y} = E_m
\]

\[
\Rightarrow E_x = E_m y + \text{const. (by integration)}
\]

4. [8 points] The electric field between the plates of a parallel-plate capacitor whose plates have a large circular radius \( R \) is given by \( E = E_m \sin \omega t \). What is the magnitude of the magnetic field between the plates of the capacitor a distance \( r < R \) from the center?

Use Maxwell’s equation:

\[
\oint_C \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 i_{\text{enc}} + \mu_0 \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{A}
\]

\[
\Rightarrow 2\pi r B = \mu_0 \varepsilon_0 \pi r^2 \frac{d}{dt} (E_m \sin \omega t)
\]

\[
\Rightarrow B = \frac{1}{2} \mu_0 \varepsilon_0 r \omega E_m \cos \omega t
\]
5. The magnetic field of a plane electromagnetic wave propagating in vacuum is described by $\mathbf{B} = B_m \sin (kx + \omega t) \hat{z}$ in SI units.

(a) [4 points] In what direction does the electromagnetic wave propagate?

*Wave moves in $-x$ direction.*

(b) [6 points] Determine the expression for the electric field without introducing any new parameters.

Since the electric and magnetic fields must be in phase, and since $\frac{E_m}{B_m} = c$

$E = -cB_m \sin (kx + \omega t) \hat{y}$

*The direction is given by the Poynting vector*

$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$

$\Rightarrow \hat{k} = \hat{E} \times \hat{B} \Rightarrow -\hat{x} = -\hat{y} \times \hat{z}$

(c) [4 points] If $\omega = 4 \times 10^{15}$ s$^{-1}$ what is the wavelength of the electromagnetic wave?

$\omega = 4 \times 10^{15}$ s$^{-1} = 2\pi f$

$\lambda f = c$

$\Rightarrow \lambda = \frac{c}{f} = \frac{2\pi c}{\omega} = 470$nm
6. [8 points] Sunlight reaching Earth has an average intensity of 1.2 kW/m². Calculate the maximum strength of the electric field assuming the sunlight is a plane wave.

The Poynting vector gives the intensity, which is:

\[ \mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \]

When averaged over a wave cycle, and substituting in that \( \frac{E}{B} = c \)

\[ I = \frac{1}{2c\mu_0} E_m^2 \]

\[ \Rightarrow E_m = \sqrt{2c\mu_0 I} = 950 \text{ V/m} \]

7. [8 points] An unpolarized beam of light is sent into a stack of 4 polarizing sheets, oriented so that the angle between the polarizing directions of adjacent sheets is 30°. What fraction of the incident intensity is transmitted by the system?

The intensity of unpolarized light that passes through one polaroid filter, no matter what the angle, is always cut in half (the average of \( \cos^2 \theta \) is \( \frac{1}{2} \)):

\[ I_1 = \frac{1}{2} I_0. \]

Once the light passes through that filter, its polarization is defined by the direction of the filter. The intensity of light passing through the second filter is then given by:

\[ I_2 = I_1 \cos^2 (\theta_2 - \theta_1) = I_1 \cos^2 (30^\circ), \]

since it is the relative angle between the two filter sheets that matters. Doing this two more times for the third and fourth filters gives:

\[ I_4 = \frac{1}{2} I_0 \cos^6 (30^\circ) = \frac{1}{2} \frac{3^3}{2^6} I_0 = \frac{27}{128} I_0 = 0.21 I_0 \]
8. [8 points] The heating of air near the surface of the road leads to the familiar phenomenon of a mirage. Suppose the index of refraction of air well above the ground is 1.00027, but near the ground there is a layer of air with an index of refraction of 1.00023. What is the maximum angle from horizontal that will lead to total internal reflection of a light ray?

Suppose the surrounding air has an index of refraction of \( n_1 = 1.00027 \), and the air near the road has \( n_2 = 1.00023 \). The critical angle for total internal reflection relative to the perpendicular to the road is:

\[
\sin \theta_{\text{crit}} = \frac{n_2}{n_1} \Rightarrow \theta_{\text{crit}} = 89.49^\circ
\]

So the angle from horizontal is 0.51° or less for reflection.

9. [6 points] A double slit interference experiment finds the third bright maximum from the center, \( \Theta = 0 \), at an angle (measured from the center of the double slit to the viewing screen) of 0.3°. If the wavelength of the monochromatic light used is 500 nm, what is the separation of the slits?

\[
\sin \theta = \frac{m \lambda}{d} \text{ for constructive interference and for integer } m.
\]

The third maximum is for \( m = 3 \), so

\[
d = 3 \frac{\lambda}{\sin \theta} = 3 \frac{500 \text{ nm}}{\sin(0.3^\circ)} = 2.86 \times 10^{-4} \text{ m}
\]
10. [8 points] In the figure shown, assume the two light waves, of wavelength 555 nm in air, are initially out of phase by 180°. The indices of refraction of the media are \( n_1 = 1.46 \) and \( n_2 = 1.72 \). What is the smallest value of \( L \) that will put the waves exactly in phase once they pass through the two media?

\[
\begin{align*}
\text{The wavelength in each material is given by: } \lambda_i &= \frac{\lambda_0}{n_i} \\
\text{The number of wavelengths in each material is given by: } N_i &= \frac{L}{\lambda_i} = \frac{L}{\lambda_0 n_i}
\end{align*}
\]

Since the two waves are a half-wavelength out of phase, we want:

\[
N_2 - N_1 = \frac{1}{2}
\]

\[
\Rightarrow \frac{L_{\text{min}}}{\lambda_0} (n_2 - n_1) = \frac{1}{2}
\]

\[
\Rightarrow L_{\text{min}} = \frac{\lambda_0}{2(n_2 - n_1)} = 1067\text{nm}
\]
11. [8 points] A thin anti-reflective coating with an index of refraction of \( n_1 = 1.4 \) is placed on a lens with an index of refraction of \( n_2 = 1.5 \). What is the minimum coating thickness needed to ensure that light of wavelength 490 nm and of perpendicular incidence will be reflected from the two surfaces of the coating with fully destructive interference? Assume that the lens+coating is in air.

There will be 2 reflections, one from the surface of the coating and one from the coating-lens surface. We want destructive interference from the two, so they should be a half-wavelength out of phase. Also, since in both reflections light travels into a medium with a higher index of refraction, there is a half-wavelength shift in phase at each boundary (so it cancels out).

Thus, we want twice the thickness to be an even multiple of wavelengths plus an extra half-wavelength (in the coating material)

\[
2t = \left( m + \frac{1}{2} \right) \frac{\lambda_n}{n} \quad m=0 \text{ for minimum thickness}
\]

\[
\Rightarrow t = \frac{\lambda_n}{4n} = \frac{490 \text{nm}}{4(1.4)} = 87.5 \text{ nm}
\]