Capacitance

Disclaimer: These lecture notes are not meant to replace the course textbook. The content may be incomplete. Some topics may be unclear. These notes are only meant to be a study aid and a supplement to your own notes. Please report any inaccuracies to the professor.

A set of conductors can store electric charge. The net charge $Q=0=q-q$, but the magnitude of charge on each conductor is $\vert q \vert$.

This charge $q$ is proportional to the potential difference between the conductors:

$$q = CV$$

$$\Delta V = V_+ - V_- \equiv V$$

The constant of proportionality between charge and potential difference is $C \equiv$ capacitance. Unit is Farad (F) $\equiv$ Coulomb/Volt.

$1 \mu F = 10^{-6}$ F

$1 p F = 10^{-12}$ F

To set up a potential difference between 2 conductors requires an electric “pump”, such as a battery (see next chapter).

A larger capacitance implies that a large charge $q$ is stored for the same potential difference $V$.

Capacitance depends only on the geometry of the conductors, not the charge $q$ or voltage $V$. We can see this through examples.
**Parallel Plate Capacitor**

Consider the top view of the 2 plates:

Create a Gaussian surface (box) that extends inside and outside one of the conductor surfaces.

Gauss’ Law \( \Rightarrow \Phi = \oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{q_{enc}}{\varepsilon_0} \)

\( \mathbf{E} = 0 \) inside a conductor

\( \mathbf{E} \cdot d\mathbf{A} = 0 \) on left/right edges

\( |\mathbf{E}| \neq 0 \) on front outside face only

\( \Rightarrow \Phi = \oint_S \mathbf{E} \cdot d\mathbf{A} = EA = \frac{q_{enc}}{\varepsilon_0} \)

\( \Rightarrow q = \varepsilon_0 EA \)

The electric potential difference between the 2 plates is given by:

\[ \Delta V = V_+ - V_- = -\int_{-}^{+} \mathbf{E} \cdot ds \quad \mathbf{E} \cdot ds = -|\mathbf{E}|ds \quad \text{opposite directions} \]

\[ \Delta V = |\mathbf{E}|d \]

\( \Rightarrow |\mathbf{E}| = \frac{\Delta V}{d} \)

So for parallel plates:

\[ q = \varepsilon_0 EA = \varepsilon_0 \frac{\Delta V}{d} A = \left( \varepsilon_0 \frac{A}{d} \right) \Delta V \]

\[ q = C \Delta V \]

\[ \Rightarrow C = \varepsilon_0 \frac{A}{d} \quad \varepsilon_0 = 8.85 \times 10^{-12} \quad \frac{C^2}{Nm^2} = 8.85 \text{ pF/m} \]
**Cylindrical Capacitor (Cable)**

Let inner conductor have radius $a$, and outer radius $b$. Take Gaussian surface as cylinder between conductors ($E=0$ inside conductors).

$$\Phi = \oint_S E \cdot dA = \frac{q_{enc}}{\varepsilon_0}$$

$$\Rightarrow E2\pi L \varepsilon_0 = q \quad E \cdot dA = 0 \text{ on cylinder ends}$$

$$\Rightarrow E = \frac{q}{2\pi \varepsilon_0 L r}$$

$$\Delta V = V_+ - V_- = -\int_E E \cdot ds \quad E \cdot ds = -|E| ds \quad \text{opposite directions, but } ds = -dr \text{ opposite again}$$

$$\Delta V = -\int_a^b |E| dr = \int_a^b E |dr = \int_a^b \frac{q}{2\pi \varepsilon_0 L r} \frac{dr}{r}$$

$$\Delta V = \frac{q}{2\pi \varepsilon_0 L} \ln \frac{b}{a}$$

$$\Rightarrow q = \left(\frac{2\pi \varepsilon_0 L}{\ln b/a}\right) \Delta V$$

$$\Rightarrow C = \left(\frac{2\pi \varepsilon_0 L}{\ln b/a}\right)$$

**Spherical Capacitor**

Let inner sphere have radius $a$, and outer radius $b$. Take Gaussian surface as sphere between conductors ($E=0$ inside conductors).

Gauss’ Law \( \Rightarrow |E| = K \frac{q}{r^2} \quad a < r < b \)

$$\Delta V = V_+ - V_- = -\int_E E \cdot ds \quad E \cdot ds = -|E| ds \quad \text{opposite directions, but } ds = -dr \text{ opposite again}$$

$$\Delta V = -\int_a^b |E| dr = \int_a^b E |dr = \int_a^b Kq \frac{dr}{r^2}$$

$$\Delta V = Kq \left(\frac{1}{r}\right) \bigg|_a^b = Kq \left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\Rightarrow q = \left(\frac{ab}{K(b-a)}\right) \Delta V$$

$$\Rightarrow C = \left(\frac{ab}{K(b-a)}\right)$$
Capacitors in Parallel

Consider \( N \) capacitors all connected in parallel to the same source of potential difference \( V \). Across each capacitor \( i \) the charge on one of the plates is: \( q_i = C_i V \)

The total charge on all the plates with the same electric potential is:

\[
Q = \sum_{i=1}^{N} q_i = \sum_{i=1}^{N} C_i V = V \sum_{i=1}^{N} C_i
\]

So we can write the equivalent capacitance \( C_{\text{equiv}} \) as:

\[
Q = C_{\text{equiv}} V
\]

\[
C_{\text{equiv}} = \sum_{i=1}^{N} C_i
\]

In other words, the equivalent capacitance of \( N \) capacitors in parallel is the sum of the individual capacitances. Considering the example of parallel plate capacitors, adding several in parallel is equivalent to extending the area of the plates. Since the capacitance is proportional to the area, it increases in direct proportion.

Capacitors in Series

For \( N \) capacitors in series, the magnitude of the charge \( q \) on each plate must be the same. Consider the electric conductor connecting any 2 capacitors, and suppose that a charge \( +q \) is on the plate of one of the capacitors the conductor is connected to. Since the conductor was originally uncharged, a charge \( -q \) must exist on the plate of the second capacitor. Now a capacitor has the same charge magnitude on each plate, so by inference we can determine that the magnitude of charge on each plate in the series of capacitor must be the same.

The potential difference across any capacitor is given by \( V_i = \frac{q}{C_i} \)

The total potential difference must add up to electric potential supplied by the battery or power supply:

\[
V = \sum_{i=1}^{N} \frac{q}{C_i} = \frac{q}{C_{\text{equiv}}}
\]

So the equivalent capacitance of capacitors connected in series is given by:

\[
\frac{1}{C_{\text{equiv}}} = \sum_{i=1}^{N} \frac{1}{C_i}
\]
The potential difference across any capacitor can be determined by:

\[ q = \frac{V}{\sum_{j=1}^{N} \frac{1}{C_j}} \]

\[ V_i = \frac{q}{C_i} \]

**Energy Stored in a Capacitor**

Let’s calculate the work required of a battery or power supply to move an infinitesimal charge \( dq' \) onto the plate of a capacitor already containing a charge \( q' \). This is the same as finding the change in the potential energy of the capacitor.

Recall that the electric potential difference across a device is equal to the potential energy difference per unit charge:

\[ \Delta V = \frac{\Delta U}{q} \]

The potential energy difference is equal to the negative of the work done by the electric field to set up the configuration, or in other words equal to the work done by the power supply or battery to move the charge (the charge must move against the direction of the electric field):

\[ W_{app} = \Delta U = q\Delta V \]

So the work done to move an infinitesimal charge \( dq' \) onto the plate of a capacitor is given by:

\[ dW_{app} = dq'\Delta V \]

If the capacitor already has a charge \( q' \), then \( \Delta V = \frac{q'}{C} \)

So \( dW_{app} = \frac{q'}{C} dq' \)

So to charge up a capacitor initially uncharged to a total charge \( q \) will require integrating over the above expression:

\[ W_{app} = \int dW_{app} = \frac{1}{C} \int_0^q q'dq' = \frac{1}{C} \frac{q^2}{2} \]

\[ W_{app} = \Delta U = \frac{q^2}{2C} \]
Since \( q = C \Delta V \) for a capacitor, the electric potential energy stored in a capacitor can be expressed in 2 ways:

\[
\Delta U = \frac{q^2}{2C} = \frac{1}{2} C (\Delta V)^2
\]

This potential energy can be used to perform work if the capacitor is disconnected from the power supply and connected to an electrical circuit. For example, a flash bulb on a camera works in this way.

Using both forms of the relation for the energy in a capacitor, we can see which capacitor has a greater energy when two are connected in series or parallel. When two capacitors are in series, each has the same charge \( q \) on one of the plates. Thus by \( \Delta U = \frac{q^2}{2C} \), the smaller capacitance has the greater energy stored. For two capacitors in parallel, both capacitors have the same voltage across the plates. Thus by \( \Delta U = \frac{1}{2} C (\Delta V)^2 \), the larger capacitance stores the greater energy.

### Energy Stored in an Electric Field

Let’s apply the expression for the potential energy to the specific example of a parallel plate capacitor with plate area \( A \) and plate separation \( V \). The capacitance is given by:

\[
C = \varepsilon_0 \frac{A}{d}
\]

The magnitude of the electric field between the plates is given by \( |E| = \frac{V}{d} \). So the potential energy stored in the capacitor is

\[
U = \frac{1}{2} C (\Delta V)^2 = \frac{\varepsilon_0 A |E|^2 d^2}{2d}
\]

and per unit volume \( V = Ad \), the energy density is given by

\[
u = \frac{1}{2} \varepsilon_0 |E|^2
\]

This result is more general—it applies to any capacitor. Even more, one can interpret the result as saying the potential energy of the capacitor is stored in the electric field of the capacitor. The electric field has a reality to it, and contains an energy density given by the
above expression. The field is able to do work on electric charges by expending this potential energy.

**Dielectrics**

If a material (matter) is inserted between the plates of a capacitor the capacitance increases:

\[ C = \kappa C_{\text{vac}} \]

where \( \kappa \) is called the dielectric constant of the material. For air, \( \kappa = 1.00054 \), other materials have higher dielectric constants.