Relativistic Kinematics I

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1 Mechanics in special relativity

1.1 Lorentz transformation: length contraction, time dilation, proper time

Space and even time coordinates have different values when measured in different inertial frames moving with respect to one another (and/or rotated). Consider two such frames in which one frame is moving with respect to the other at a velocity \( v \) along the +z axis. Let coordinates measured in the moving frame be denoted with a prime. Then the Lorentz Transformation (LT) describes how the coordinates measured in the two frames are related.

\[
\begin{align*}
x' &= x \\
y' &= y \\
z' &= \gamma(z - \beta ct) \\
ct' &= \gamma(ct - \beta z)
\end{align*}
\]

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where $\beta = v / c$ and $\gamma = 1 / \sqrt{1 - \beta^2}$. I used $ct$ instead of $t$ to maintain length units everywhere. Note that the equations are symmetric (they can be derived from one another) and differ only by the sign of the frame velocity. The LT preserves the quantity $(ct)^2 - x^2 - y^2 - z^2$, discussed in detail later. By comparison, in Newtonian mechanics $\gamma = 1$ and $t' = t$.

Several comments should be made.

- Lengths perpendicular to the direction of motion are unaffected by LTs.
- The length of an object along its direction of motion is related to its length at rest (primed frame) as $\Delta z = \Delta z' / \gamma$, i.e. length contraction.
- The measured time interval of a moving clock is related to the elapsed time at rest (primed frame) by $\Delta t = \gamma \Delta t'$, i.e. time dilation. We call a time interval for an object at rest the “proper time” $\tau$. Thus $t = \gamma \tau$ and $t \geq \tau$. The proper time concept is useful when comparing the lifetime of a particle in its rest frame and in the frame in which it is moving.

### 1.2 Relativistic energy and momentum

For a particle with velocity $u$, the relativistic generalization of momentum is $p = \gamma_u m u$, where $\gamma_u = 1 / \sqrt{1 - u^2 / c^2}$. Force is defined as before: $F = dp / dt$. The relativistic energy (total energy) is $E = \gamma_u m c^2$, which can be derived from the familiar definition

$$\Delta E = \int_{x_1}^{x_2} \frac{dp}{dt} \cdot dx = \int_{u_1}^{u_2} \frac{dE}{du} \cdot u = \int_{u_1}^{u_2} \left(1 - u^2 / c^2\right)^{-3/2} u \, du = \gamma_u m c^2 - \gamma_u mc^2$$

The total energy has the expansion $E = mc^2 + \frac{1}{2} m u^2 + \frac{3}{8} m u^2 (u / c)^2 \cdots$ where the first term is the “mass energy”, the second is the familiar Newtonian expression for kinetic energy and the others are relativistic corrections. For ordinary velocities, the third and higher terms are negligible. For a velocity of 100 km/hour, for example, the relativistic correction is $3 \times 10^{-15}$ times the kinetic energy term.

Velocity can easily be obtained from the energy and momentum using $u = pc^2 / E$.

The relativistic momentum can be derived using the analog to the Newtonian formula but using proper time instead

$$p = m \frac{dx}{d\tau} = m \frac{dx}{dt} \frac{dt}{d\tau} = \gamma_u m u$$

$$E = mc^2 \frac{dt}{d\tau} = \gamma_u mc^2$$

Since the proper time $\tau$ is a scalar, the energy and momentum transform the same way as time and length coordinates under an LT. Using the same coordinate frames as before, and using consistent momentum units, we get the identical transformations.
\[ p'_x = p_x \quad p_x = p'_x \]
\[ p'_y = p_y \quad p_y = p'_y \]
\[ p'_z = \gamma \left( p_z - \beta E / c \right) \quad p_z = \gamma \left( p'_z + \beta E / c \right) \]
\[ E' / c = \gamma \left( E / c - \beta p_z \right) \quad E / c = \gamma \left( E' / c + \beta p_z \right) \]

1.3 Natural units

When doing kinematics we will use natural units where energy, momenta and masses are all expressed in GeV (1 GeV = \( 10^9 \times \left( 1.6 \times 10^{-19} \right) = 1.6 \times 10^{-10} \) J) and velocities are expressed in units of \( c \) (i.e., \( c = 1 \)). However, you will always see published papers use masses and momenta expressed as GeV/c^2 and GeV/c, respectively. These relations are strictly true regardless of units. From the LT formulas it’s easy to see that the values of energy, momentum and mass form a right triangle (see Figure 1), with \( E^2 = \left( pc \right)^2 + \left( mc^2 \right)^2 \) or \( E^2 = p^2 + m^2 \) in natural units (\( c = 1 \)). The velocity is \( u = pc^2 / E \) or \( u = p / E \) in natural units.

![Figure 1: Relativistic “triangle relation” between mass, momentum and energy](image)

1.4 Examples

Example 1: Find the relativistic energy and momentum of a \( Ks \) (“K-short”) meson (\( m_{Ks} = 0.4977 \) GeV) moving along the +z axis at \( u_{Ks} = 0.95c \) in some coordinate system. We use natural units.

\[
\gamma_{Ks} = \frac{1}{\sqrt{1 - 0.95^2}} = 3.202563 \\
E_{Ks} = \gamma_{Ks} m_{Ks} = 3.202563 \times 0.4977 = 1.593916 \text{ GeV} \\
p_{xKs} = 0 \\
p_{zKs} = 0 \\
p_{xKs} = \gamma_{Ks} m_{Ks} v_{Ks} = 3.202563 \times 0.4977 \times 0.95 = 1.514220 \text{ GeV} 
\]

To convert to SI units, we write the mass as \( m_{Ks} = 0.4977 \text{ GeV} / c^2 \) and momentum as \( p_{Ks} = 1.514220 \text{ GeV} / c \) and then convert using the value of GeV in joules and \( c \) in m/s. Note
that if the energy and momentum are measured, the mass can be calculated from
\[ m = \sqrt{E^2 - p^2} = \sqrt{1.593916^2 - 1.514220^2} = 0.4977 \text{ GeV}. \]

In relativistic mechanics a massless particle has \( E = p \), thus massless particles can carry energy and momentum, unlike in Newtonian physics. The speed of a massless particle is always \( c \) \((u = p/E = 1)\). When a particle is extremely relativistic \((E \gg m)\), then \( E = p \) and \( u = 1 \), so it behaves essentially like a massless particle. The Ks example above is highly relativistic since its momentum is approximately three times the mass in natural units. Most particles produced in collisions at the LHC are highly relativistic.

**Example 2**: Consider a photon with energy 2.5 GeV moving along the \(+x\) axis. Since photons are massless, we have \( E = p \) in natural units. The photon 4-momentum is \((2.5, 2.5, 0, 0)\) and its velocity is \( u = 1 \). The gamma factor is infinite but is not used anyway.

## 2 4-Vectors and Lorentz Transformations

### 2.1 3-Vectors and rotations

Position \((\vec{x})\) and momentum \((\vec{p})\) are examples of 3-vectors with components that can be measured in different coordinate systems. However, the quantities \( \vec{x}^2 = \vec{x} \cdot \vec{x} = x^2 + y^2 + z^2 \) and \( \vec{p}^2 = \vec{p} \cdot \vec{p} = p_x^2 + p_y^2 + p_z^2 \) are invariant under rotations. For any vector \( \vec{A} \), we call
\[
A = \sqrt{A_1^2 + A_2^2 + A_3^2} \quad \text{the magnitude of } \vec{A}, \text{ with } A \geq 0.
\]
We use Latin indices \((A_i, A_j, A_k, \text{ etc. where } i, j, k \text{ run over } 1, 2, 3)\) to denote the components of \( \vec{A} \).

If \( \vec{x} \) has components \((x, y, z)\) in one coordinate system, its components \((x', y', z')\) in a coordinate system that is rotated around the \(z\) axis by an angle \( \theta \) (counterclockwise) are calculated from
\[
x' = x \cos \theta + y \sin \theta \\
y' = -x \sin \theta + y \cos \theta \\
z' = z
\]

Other kinds of rotations give similar equations. Note that any rotation can be specified in general by three independent quantities (e.g., Euler angles). We usually write the transformation as \( \vec{A}' = R \vec{A} \), where \( R \) is a \(3 \times 3\) matrix satisfying \( R^T R = I \) and \( \det(R) = 1 \). In the above rotation, for example:
\[
R = \begin{pmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{pmatrix} = \begin{pmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

For general rotations the matrix is more complicated. For all vectors \( \vec{A} \) and \( \vec{B} \), however, rotations preserve the “dot product” \( \vec{A} \cdot \vec{B} \equiv A_x B_x + A_y B_y + A_z B_z \).
2.2 4-Vectors

In relativistic mechanics, the interesting quantities are 4-vectors, where \( x = (ct, \mathbf{x}) \) and \( p = (E/c, \mathbf{p}) \) are the space-time and energy-momentum 4-vectors, respectively.

A 4-vector \( A \) is written as \( A = (A_0, \mathbf{A}) \) and the indices are specified by Greek letters. Thus we write the energy-momentum vector components as \( p_\mu \), where \( p_0 = E/c \), \( p_1 = p_x \), \( p_2 = p_y \) and \( p_3 = p_z \) (we will set \( c = 1 \) later). The linearity of the Lorentz transformation guarantees that linear combinations of 4-vectors \( (a_1 p_1 + a_2 p_2 + \cdots) \), where the \( a_i \) are constants) are also 4-vectors.

In analogy with the space-time 4-vector, we sometimes refer to the timelike (0\(^{th}\)) and spacelike components.

Other examples of 4-vectors are the electromagnetic potential \( \phi / c, \mathbf{A} \), the electromagnetic current \( \rho c \mathbf{j} \) and the wave vector \( (\omega / c, \mathbf{k}) \).

2.3 Lorentz transformation and boost

If the components of a 4-vector are measured in one coordinate system, their values in another coordinate system rotated or moving relative to the first can be calculated via the Lorentz Transformation (LT), as discussed in Section 1.1. The quantity \( A^2 = A_0^2 - \mathbf{A}^2 \) is invariant under all LTs. This is similar to rotations preserving the value \( \mathbf{A}^2 \). For any two 4-vectors \( A \) and \( B \), the dot product \( A \cdot B = A_0 B_0 - \mathbf{A} \cdot \mathbf{B} \) is invariant under an LT. If \( A^2 > 0 \) then we say that the 4-vector is “timelike” (timelike component is larger than the spacelike components), if negative it is “spacelike”.

A Lorentz Transformation between two frames is in general a 4 \times 4 matrix specified by 6 independent quantities, three velocities (specifying a “boost” along some direction) and three angles (specifying a rotation). We mainly consider boosts in this course.

2.4 Boost along the z direction

We discussed in Section 1.1 how coordinates and energy-momentum transform between two frames with one frame moving at a velocity \( v \) along the +z axis (in particle physics we prefer to use motion along or rotation around the \( z \) axis instead of the \( x \) axis). We found

\[
\begin{align*}
ct' &= \gamma_v (ct - \beta_v z) \\
x' &= x \\
y' &= y \\
z' &= \gamma_v (z - \beta_v ct) \\
\end{align*}
\]

\[
\begin{align*}
E'/c &= \gamma_v (E/c - \beta_v p_z) \\
p'_x &= p_x \\
p'_y &= p_y \\
p'_z &= \gamma_v (p_z - \beta_v E/c) \\
\end{align*}
\]

where \( \beta_v = v/c \) and \( \gamma_v = 1/\sqrt{1-\beta^2_v} \). The transformation of a 4-vector \( A \) to its value \( A' \) in a “boosted” frame moving at velocity \( v \) along the +z axis can thus be written \( A' = \Lambda_v A \) where \( \Lambda_v \) is a 4 \times 4 matrix:
\[
\Lambda_v = \begin{pmatrix}
\gamma_v & 0 & 0 & -\beta_v \gamma_v \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\beta_v \gamma_v & 0 & 0 & \gamma_v
\end{pmatrix}
\]

You can verify that the old and new components satisfy \( A'^2 = A^2 \) or \( A'^2 - A^2 = A_0^2 - A^2 \).

From now on we use \( c = 1 \) (natural) units so that momentum and energy are expressed in units of GeV and velocities are relative to \( c \).

**Example:** Let’s find the \( K_s \) energy-momentum components from the example in Section 1.4 in a second coordinate system moving at a velocity 0.6\( c \) along the +z axis. Thus \( v = +0.6 \) and \( \gamma_v = 1/\sqrt{1 - 0.6^2} = 1.25 \). The initial 4-momentum is (1.593916, 0, 0, 1.514220). The components in the new frame are

\[
\begin{align*}
E'_{Ks} &= \gamma_v \left( E_{Ks} - \nu p_z_{Ks} \right) = 1.25 \left( 1.593916 - 0.6 \times 1.514220 \right) = 0.856730 \\
p'_{xKs} &= p_{xKs} = 0 \\
p'_{yKs} &= p_{yKs} = 0 \\
p'_{zKs} &= \gamma_v \left( p_z_{Ks} - \nu E_{Ks} \right) = 1.25 \left( 1.514220 - 0.6 \times 1.593916 \right) = 0.697338
\end{align*}
\]

Note that \( E'^2_{Ks} - p'^2_{Ks} = E^2_{Ks} - p^2_{Ks} = 0.247705 = m^2_{Ks} \), as expected. The matrix for the transformation \( p'_{Ks} = \Lambda_v p_{Ks} \) can be written

\[
\Lambda_v = \begin{pmatrix}
\gamma_v & 0 & 0 & -\beta_v \gamma_v \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\beta_v \gamma_v & 0 & 0 & \gamma_v
\end{pmatrix} = \begin{pmatrix}
1.25 & 0 & 0 & -0.75 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-0.75 & 0 & 0 & 1.25
\end{pmatrix}
\]

### 2.5 General properties of the Lorentz transformation

A general LT is specified by 6 independent parameters, which can be taken to be 3 angles and 3 velocities. We will only consider boosts in this course. It is relatively straightforward to generalize the previous LT formula for a boost along the +z axis to a boost along an arbitrary direction by using components parallel and perpendicular to the boost direction. Can you see how? This is the basis for computer functions that transform the 4-momentum of a particle from one frame to another. We can derive the appropriate formulas in a few simple steps, as will be shown in the next note on relativistic kinematics.

### 3 Lorentz Transformation of Velocities and Angles

The transformation of angles and velocities can be found easily from the momentum components.
3.1 Lorentz transformation of velocity

From Section 1.2, we saw that the particle velocity \( \mathbf{u} \) is \( \mathbf{u} = \mathbf{p} / E \). The particle velocity \( \mathbf{u}' \) in a frame moving with velocity \( \mathbf{v} \) along the +z axis is:

\[
\begin{align*}
    u'_x &\equiv \frac{dx'}{dt'} = \frac{dx}{\gamma_v (dt - vdz)} = \frac{u_x}{\gamma_v (1 - vu_z)} \\
    u'_y &\equiv \frac{dy'}{dt'} = \frac{dy}{\gamma_v (dt - vdz)} = \frac{u_y}{\gamma_v (1 - vu_z)} \\
    u'_z &\equiv \frac{dz'}{dt'} = \frac{\gamma_v (dz - vdt)}{\gamma_v (dt - vdz)} = \frac{u_z - v}{1 - vu_z}
\end{align*}
\]

It is straightforward to prove that the components \( u'_i \) calculated in this way satisfy \( u'_i = p'_i / E' \).

3.2 Lorentz transformation of angle

Let \( \theta \) be the angle of the particle motion relative to the \( z \) axis in a coordinate frame. Then \( p_\perp = p \sin \theta \) is the momentum component perpendicular to the \( z \) axis \( (p^2_\perp = p^2_x + p^2_y) \) and \( p_z = p \cos \theta \) is the component along \( z \). Similar relations hold in the primed (moving) frame. Note that \( p'_\perp = p_\perp \) because LTs leave the momenta perpendicular to the direction alone. The transformed \( \theta \) is determined from

\[
\tan \theta' = \frac{p'_\perp}{p'_z} = \frac{p_\perp}{\gamma_v (p_z - vE)} = \frac{p \sin \theta}{\gamma_v (p \cos \theta - vE)} = \frac{\sin \theta}{\gamma_v (\cos \theta - v / u)}
\]

where \( \theta \) and \( u = \mathbf{p} / E \) are the angle and velocity of the particle in the first frame, respectively.

3.3 CM to Lab Lorentz transformation

It’s very common to boost particles resulting from a collision or decay process from a CM system to the lab frame. This requires that we switch primed and unprimed quantities and replace \( \mathbf{v} \) by \( -\mathbf{v} \) in the above formulas. Using the convention that CM values are marked with an asterisk, we obtain:

\[
\begin{align*}
    u'_x &\equiv \frac{dx}{dt} = \frac{dx^*}{\gamma_v (dt^* + vdz^*)} = \frac{u^*_x}{\gamma_v (1 + vu^*_z)} \\
    u'_y &\equiv \frac{dy}{dt} = \frac{dy^*}{\gamma_v (dt^* + vdz^*)} = \frac{u^*_y}{\gamma_v (1 + vu^*_z)} \\
    u'_z &\equiv \frac{dz}{dt} = \frac{\gamma_v (dz^* + vdt^*)}{\gamma_v (dt^* + vdz^*)} = \frac{u^*_z + v}{1 + vu^*_z}
\end{align*}
\]
\[
\tan \theta = \frac{p_\perp}{p_z} = \frac{p^*_\perp}{\gamma_v \left( p^*_z + \nu E^* \right)} = \frac{p^* \sin \theta^*}{\gamma_v \left( p^* \cos \theta^* + \nu E^* \right)} = \frac{\sin \theta^*}{\gamma_v \left( \cos \theta^* + v / u^* \right)}
\]

For a process or decay taking place in a highly relativistic moving system, the angles that particles make relative to the frame’s direction of motion are shrunk (“folded”) by a factor \( \sim 1 / \gamma_v \).

The only exception occurs when a particle is emitted almost backwards (\( \cos \theta^* \approx -1 \)) in the moving frame.

### 3.4 Example of angle folding: \( \pi^0 \rightarrow \gamma \gamma \) at high energy

Consider a 50 GeV \( \pi^0 \) (\( m_\pi = 0.135 \text{ GeV} \)) undergoing the decay \( \pi^0 \rightarrow \gamma \gamma \). In the \( \pi^0 \) rest frame, the photons are emitted at arbitrary angles (but opposite directions). But in the frame where the \( \pi^0 \) is moving, where \( \gamma = 50 / 0.135 = 370 \), the photon angles relative to the \( \pi^0 \) direction of motion are shrunk by a factor \( \sim 370 \) or \( \sim 2-3 \text{ mr} \), making the separation of the photons so small that a highly segmented calorimeter is needed to detect the individual photons.

### 4 Proper Time, Time Dilation and Decays of Moving Particles

As discussed in Section 1.1, if a particle experiences an elapsed time \( \tau \) in its rest frame (“proper time”), the elapsed time in a frame where the particle is moving with velocity \( u \) is \( t = \gamma_u \tau \). Thus elapsed time in any coordinate system where the particle moves is always longer (“dilated”) than the proper time. Time dilation has the practical consequence that a particle with a lifetime \( \tau \) in its rest frame has a laboratory lifetime of \( \gamma_u \tau \) and average decay distance \( \gamma_u v \tau = c \tau \left( p / m \right) \) for momentum and mass measured in GeV (this is why the PDG publishes values of \( c \tau \) as well as \( \tau \)).

#### 4.1 Example 1: Average muon decay distance

A muon (mass 106 MeV, lifetime 2.2\( \mu \text{s} \)) has a momentum of 15 GeV. What is the average distance the muon moves before it decays? The average distance moved is \( v \gamma \tau \), or \( c \tau p / m \) using GeV units for momentum and mass and \( p / m = v \gamma \). For the parameters shown we find that the muon travels an average distance of \( \left( 3 \times 10^8 \right) \times \left( 2.2 \times 10^{-6} \right) \times \left( 15 / 0.106 \right) = 93.4 \text{ km} \). Without the time dilation factor the muon would only travel \( \left( 3 \times 10^8 \right) \times \left( 2.2 \times 10^{-6} \right) = 660 \text{ m} \) and cosmic ray muons produced in the upper atmosphere would rarely reach the earth’s surface.

#### 4.2 Example 2: Unstable particles used in a beam

Physicists frequently create beams of particles such as neutrinos, muons, charged pions and kaons and “hyperons” (baryons with strangeness such as \( \Lambda \) and \( \Sigma^\pm \)). Since most of these particles are unstable, there are challenges involved in having these particles reach a target before they decay.
For example, suppose we create a charged pion beam (PDG values $m_\pi = 0.1396$ GeV, $c\tau_\pi = 7.80$ m) with $E_\pi = 24.0$ GeV and send it to a target a distance 2.5 km away. What fraction of the original beam will make it to the target?

The distance traveled in an average lifetime is $c\tau_\pi \left( p_\pi / m_\pi \right)$. Since the beam is extremely relativistic, $E_\pi = p_\pi$ and the average distance traveled is $7.80 \times \left( 24 / 0.140 \right) = 1340$ m. The fraction of pions making it to the target is thus $e^{-2500/1340} = 15\%$.

Similar considerations apply to the design of a “muon collider”, in which muons are accelerated to TeV energies in a circular synchrotron. The acceleration time must be short enough so that a significant fraction of muons survive the multi-loop acceleration process. Additional information on muon colliders is posted at http://www.fnal.gov/pub/muon_collider/.

4.3 Example 3: Cosmic ray spectrum

Suppose cosmic ray muons are created in the upper atmosphere at an altitude of 10 km with an energy spectrum given by $dN / dE_\mu \propto e^{-0.4 E_\mu}$. Ignoring interactions with the atmosphere, what is the energy spectrum of the muons that make it to the earth’s surface? Plot the original spectrum and the final spectrum using wxMaxima using [logy] in the wxPlot2d command for muon energies from 1 to 20 GeV.

5 References

1 For a good introduction to relativity, see the lectures by Darin Acosta:
www.phys.ufl.edu/~acosta/phy2061/lectures/Relativity2.pdf
www.phys.ufl.edu/~acosta/phy2061/lectures/Relativity3.pdf
www.phys.ufl.edu/~acosta/phy2061/lectures/Relativity4.pdf