1. In about three sentences discuss the possibility that a unit of electric charge could disappear at one location, and simultaneously reappear somewhere else, thereby conserving electric charge.

Answer:

(a) If a charge disappears from one location and simultaneously reappears at a different location, then charge is conserved only in that particular frame of reference. Any other frame of reference, would see the disappearance and reappearance as not simultaneous, and charge would not be conserved.
2. Three Lorentz boosts with velocities $v_1$, $v_2$ and $v_3$ are applied successively in the same direction.

(a) What is the velocity $v$ of the net transformation?

(b) Does the order in which they are applied matter?

(c) Next, the boost $v_1$ is applied in the $x$-direction, $v_2$ in the $y$-direction and $v_3$ in the $z$ direction. Does the order matter now?

Answer:

(a) There are several ways to do this:

(i) Repeated use of the velocity addition formula gives

$$v = \left[ \frac{(v_1 + v_2)/(1 + v_1v_2)}{1 + [(v_1 + v_2)/(1 + v_1v_2)]v_3} \right] + v_3 = \frac{v_1 + v_2 + v_3 + v_1v_2v_3}{1 + v_1v_2 + v_1v_3 + v_2v_3}.

(ii) Multiplying transformation matrices gives

$$\begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix} = \begin{pmatrix} \gamma_3 \gamma v_3 & \gamma_2 \gamma v_2 \\ \gamma_3 v_3 & \gamma_3 \end{pmatrix} \begin{pmatrix} \gamma_1 \gamma v_1 \\ \gamma_2 \gamma v_2 \end{pmatrix} = \gamma_1 \gamma_2 \gamma_3 \begin{pmatrix} \gamma_1 v_1 + v_1 v_2 + v_1 v_3 + v_2 v_3 & v_1 + v_2 + v_3 + v_1v_2v_3 \\ v_1 + v_2 + v_3 + v_1v_2v_3 & 1 + v_1v_2 + v_1v_3 + v_2v_3 \end{pmatrix}.

Comparing corresponding entries gives the same result. In both cases, the symmetry of the result means that it is impossible to tell what order the boosts are applied.

(b) Direct computation gives

$$B_z(v_3)B_y(v_2)B_x(v_1) = \begin{pmatrix} \gamma_1 \gamma_2 \gamma_3 & \gamma_1 \gamma_2 \gamma_3 v_1 & \gamma_2 \gamma_3 v_2 & \gamma_3 v_3 \\ \gamma_1 v_1 & \gamma_1 & 0 & 0 \\ \gamma_1 \gamma_2 v_2 & \gamma_1 \gamma_2 v_1 v_2 & \gamma_2 & 0 \\ \gamma_1 \gamma_2 \gamma_3 v_3 & \gamma_1 \gamma_2 \gamma_3 v_1 v_3 & \gamma_2 \gamma_3 v_2 v_3 & \gamma_3 \end{pmatrix}

but

$$B_z(v_3)B_x(v_1)B_y(v_2) = \begin{pmatrix} \gamma_1 \gamma_2 \gamma_3 & \gamma_1 \gamma_3 v_1 & \gamma_1 \gamma_2 \gamma_3 v_2 & \gamma_3 v_3 \\ \gamma_1 \gamma_2 v_1 & \gamma_1 & \gamma_1 \gamma_2 v_1 v_2 & 0 \\ \gamma_2 v_2 & 0 & \gamma_2 & 0 \\ \gamma_1 \gamma_2 \gamma_3 v_3 & \gamma_1 \gamma_3 v_1 v_3 & \gamma_1 \gamma_2 \gamma_3 v_2 v_3 & \gamma_3 \end{pmatrix},

which are visibly different (but have a certain pattern exchanging 1 and 2).
3. In a high energy accelerator, the energy available to create new particles is the energy in the center of mass frame.

(a) Consider a proton with momentum 200GeV/c incident on a target proton at rest. What is the available energy in the center of mass frame?
(b) Next, consider a 200GeV proton heading east colliding with a 200GeV proton headed west. What is the available energy in the center of mass frame now?
(c) What momentum would be needed in a fixed-target experiment to obtain the same available energy? (This has something to do with why most large accelerators are now colliders.)
(d) A $\Lambda^0$ baryon ($m_\Lambda = 1115.7$MeV) decays into a proton ($m_p = 938.3$MeV) and a negative pion ($m_\pi = 139.6$MeV). What is the momentum of the proton or pion in the center of mass frame?
(e) The decaying $\Lambda^0$ has momentum 28.5GeV/c in the lab frame. What is the maximum angle between the proton and the pion in the lab?

Answer:

(a) The center-of-mass frame is the frame where the total spatial momentum vanishes, $P^\mu = p_1^\mu + p_2^\mu = (E_{cm}, 0)$, so $P \cdot P = -E_{cm}^2$. Again, $P \cdot P$ is an invariant that can be evaluated in any frame, including the lab frame. For the fixed target, $p_1^\mu = (E, p)$, $p_2^\mu = (m, 0)$, and

$$E_{cm}^2 = (E + m)^2 - |p|^2 = E^2 + m^2 + 2mE - |p|^2 = 2m(E + m).$$

For $E = 200$GeV, this gives $E_{cm} = 20$GeV.

(b) For colliding beams, $p_1^\mu = (E_1, p_1)$, $p_2^\mu = (E_2, -p_2)$, and

$$E_{cm}^2 = (E_1 + E_2)^2 - (p_1 + p_2)^2 = (2E)^2 - 0^2 = 4E^2,$$

or $E_{cm} = 2E = 400$GeV (here the lab frame is the center-of-mass frame).

(c) With a fixed target, to obtain this $E_{cm}$ requires a beam energy $E = E_{cm}^2/2m = 80$TeV.

(d) Conservation of energy, $M_\Lambda = E_p + E_\pi$, gives

$$p^2 = \frac{(M_\Lambda^2 - m_p^2 - m_\pi^2)^2 - 4m_p^2m_\pi^2}{2M_\Lambda^2}$$

Numerically, this is $p = 100.53$MeV. Check: $E_p = \sqrt{p^2 + m_p^2} = 943.67$MeV; $E_\pi = \sqrt{p^2 + m_\pi^2} = 172.03$MeV; $E_p + E_\pi = 1115.70$MeV = $M_\Lambda$.

(e) The energy and momentum of the proton and the pion in the center of mass frame are known. For an angle $\phi$ from the beam direction in the center of mass frame, the components of $\vec{p}$ in the lab are

$$p_{p,x} = \gamma(vE_p + p\cos \phi) \quad p_{p,y} = p\sin \phi$$
$$p_{\pi,x} = \gamma(vE_\pi - p\cos \phi) \quad p_{\pi,y} = -p\sin \phi$$
where $\gamma v M = 28.5$ GeV. In the lab, the direction of the beam, $\vec{p}_p$ and $\vec{p}_\pi$ are coplanar, so the opening angle is the sum of the angles between each $\vec{p}$ and the beam:

$$\theta = \tan^{-1} \left[ \frac{p \sin \phi}{\gamma (v E_p + p \cos \phi)} \right] + \tan^{-1} \left[ \frac{p \sin \phi}{\gamma (v E_\pi - p \cos \phi)} \right].$$

The maximum opening angle is then obtained for $\phi = 1.006$ rad,

$$\theta = 0.0315 \text{ rad} = 1.80^\circ$$
4. In the frame of reference $O$, the 4-velocity of a rocket ship is $u^a = (2, 1, 1, 1)$ (note that the units assume that $c = 1$). The rocket encounters a high velocity cosmic ray whose momentum is $P^a = (300, 299, 0, 0) \times 10^{-27}$ kg. Compute the energy of the cosmic ray as measured by the rocket ship’s passengers, using each of the two following methods.

(a) Find the Lorentz transformations from $O$ to the Momentarily Comoving Reference Frame of the rocket and use it to transform the components of $P^a$.

(b) Find the answer by contracting $u^a$ into $P_a$.

(c) Which method is easier?

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**Answer:**

(a) The answer is unique only up to the possibility of a spatial rotation after the proper Lorentz transformation. There is no particularly easy way to do this. What appears easiest to me while looking at this problem set is to use the answer to problem (2.b), and solve

\[
\begin{pmatrix}
\gamma_1 \gamma_2 \gamma_3 v_1 \\
\gamma_1 v_1 \\
\gamma_1 \gamma_2 v_2 \\
\gamma_1 \gamma_2 \gamma_3 v_3
\end{pmatrix}
\begin{pmatrix}
\gamma_1 \gamma_2 \gamma_3 v_1 \\
\gamma_2 \gamma_3 v_2 \\
\gamma_3 v_3 \\
1
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
1 \\
1 \\
1
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix}
\]

for $v_1, v_2, \text{ and } v_3$. And the easiest way for me is to use the following maple script, given below, just before problem 5.

(b) Use $E = -u_a P^a = - (-600 + 299) \times 10^{-27} \text{ kg} = 301 \times 10^{-27} \text{ kg}$.

(c) The second way is much easier!
Maple script: for problem (4.a)

```maple
$ maple
                      Maple V Release 5.1 (WMI MVR5.1 Campus Wide License)
                      Copyright (c) 1981-1998 by Waterloo Maple Inc. All rights
                      reserved. Maple and Maple V are registered trademarks of
                      Waterloo Maple Inc.
                      Type ? for help.
> g1 := 1/(1-v1^2)^(1/2);
> g2 := 1/(1-v2^2)^(1/2);
> g3 := 1/(1-v3^2)^(1/2);
> eqs := {2*g1*v1 + g1 = 0,
         2*g1*g2*v2 + g1*g2*v1*v2 + g2 = 0,
         2*g1*g2*g3*v3 + g1*g2*g3*v1*v3 + g2*g3*v2*v3 + g3 = 0}:
> ans := solve(eqs,{v1,v2,v3});
  1/2  1/2
ans := {v2 = - 1/3 3 , v3 = - 1/2 2 , v1 = -1/2}
> simplify(subs(ans, energy = g1*g2*g3*300 + g1*g2*g3*v1*299));
  energy = 301
```
5. An observer with a four-velocity $u^a$, observes an asteroid whose four-velocity is $A^a$. Find a covariant expression for the three-velocity of the asteroid as measured by the observer. Do this for part of the next assignment.