1. Given a tensor $F_{ab} = F_{[ab]}$, show that

$$F_{c}^{b} \nabla_{b} F_{a}^{c} = -F_{bc} \nabla_{c} F_{ab}.$$  

**Ans:** $F_{bc} \nabla_{b} F_{ac} = F_{cb} \nabla_{c} F_{ab} = -F_{bc} \nabla_{c} F_{ab}$

2. The Schwarzschild geometry in Schwarzschild’s original coordinates $(t, r, \theta, \phi)$ is

$$g_{ab} dx^{a} dx^{b} = -\left(1 - \frac{2m}{r}\right) dt^{2} + \left(1 - \frac{2m}{r}\right)^{-1} dr^{2} + r^{2} d\theta^{2} + r^{2} \sin^{2} \theta d\phi^{2}.$$  

where $m$ is a constant which is interpreted as the mass of the black hole. An object of very small mass is held at a fixed spatial position $(r, \theta, \phi)$.

(a) What are the components of the four-velocity $u^{a}$ of the object? (Assume, as always, that the four velocity is properly normalized.)

**Ans:** Only the $u^{t}$ component is non-zero. Thus, from the normalization,$$u^{t} = (1 - 2m/r)^{-1/2}.$$  

(b) What are the components of the four acceleration $a^{b} = u^{a} \nabla_{a} u^{b}$ of the object? Consideration of this answer when $r \gg m$ reveals why the constant $m$ is referred to as the “mass” of the black hole.

**Ans:** While the object is being held, it is not moving along a geodesic; therefore, it is accelerating. The covariant components of acceleration are

$$a_{b} = u^{a} \nabla_{a} u_{b} = \frac{du_{b}}{d\tau} - \frac{1}{2} u^{a} u^{c} \frac{\partial g_{ac}}{\partial x^{b}}.$$  

Thus the only nonzero component of acceleration is

$$a_{r} = \frac{du_{r}}{d\tau} - \frac{1}{2} u^{a} u^{c} \frac{\partial g_{ac}}{\partial r}$$

$$= 0 - \frac{1}{2} (1 - 2m/r)^{-1} \frac{\partial g_{tt}}{\partial r}$$

$$= \frac{1}{2} (1 - 2m/r)^{-1} \frac{2m}{r^{2}}$$

$$= (1 - 2m/r)^{-1} \frac{m}{r^{2}}.$$
And the contravariant component is

\[ a^r = \frac{m}{r^2}. \]

Note that \( a^b \) points straight up in the radial direction as long as the particle is at rest.

3. A small particle moves in flat space in the \( x \) direction, along a timelike, worldline (not a geodesic!) which is parameterized by its proper time \( \tau \), so that \( u^a u_a = -1 \). The particle is at rest for \( t < 0 \), and has coordinate speed

\[ \frac{dx}{dt} = \sqrt{1 - e^{-2t/T}} \text{ for } t \geq 0 \]

for some constant \( T \).

(a) Is the coordinate speed of the particle \( dx/dt \) less than the speed of light for all finite \( t \)?

**Ans:** Yes, \( \sqrt{1 - e^{-2t/T}} < 1 \) for finite \( t \).

(b) Find \( dt/d\tau \) for the particle. Hint: \( -d\tau^2 = -dt^2 + dx^2 + dy^2 + dz^2 \) in flat spacetime.

**Ans:**

\[ -d\tau^2 = -dt^2 + dx^2 = -dt^2 + \frac{dx^2}{dt} dt^2 = - \left[ 1 - \left( \frac{dx}{dt} \right)^2 \right] dt^2 = -e^{-2t/T} dt^2, \]

and

\[ \frac{dt}{d\tau} = e^{t/T}. \]

(c) Find \( dx/d\tau \) for the particle.

**Ans:**

\[ \frac{dx}{d\tau} = \frac{dx}{dt} \frac{dt}{d\tau} = e^{t/T} \sqrt{1 - e^{-2t/T}} = \sqrt{e^{2t/T} - 1}. \]

(d) What is the \( t \) coordinate of the particle when its proper time is \( \tau = T \)? Assume that \( \tau = 0 \) for the particle when \( t = 0 \).

**Ans:**

\[ \tau = \int_0^T d\tau = \int_0^t e^{-t/T} dt = T(1 - e^{-t/T}). \]

Solving this for \( t \) yields

\[ t = -T \ln(1 - \tau/T), \]

and as \( \tau \to T, \ t \) diverges logarithmically.
(e) What is the $x$ coordinate of the particle when its proper time is $\tau = T$?

**Ans:** From part (d), $e^{2t/T} = T^2/(T - \tau)^2$, and from part (c)

$$\frac{dx}{d\tau} = \sqrt{e^{2t/T} - 1} = \sqrt{T^2/(T - \tau)^2 - 1} = \frac{\sqrt{\tau(2T - \tau)}}{T - \tau}.$$  

This may be integrated with, say, Maple with the result that

$$x = -\sqrt{2T - t^2} + T \arctanh\left[\sqrt{\frac{t(2T - t)}{T^2}}\right],$$

and as $\tau \to T$, $x$ diverges logarithmically.

4. A plane gravitational wave propagating through space in the $\hat{z}$ direction is described by the metric

$$ds^2 = -dt^2 + (1 + h)dx^2 + (1 - h)dy^2 + dz^2,$$

where the function $h = h(t - z)$ is very small and represents the waveform of the gravitational wave. Assume that $h(t - z) = 0$ when $t < z$. While $t < 0$, two small mirrors float in space at rest at $(x, y, z)$ coordinates $(-\ell/2, 0, 0)$ and $(\ell/2, 0, 0)$. The mirrors are in free-fall and, therefore, move along geodesics. In this problem, assume that $h \ll 1$, and you are encouraged to work only through first order in $h$.

(a) Find three independent Killing vectors of this geometry. Hint: If two of the coordinates were $u = t - z$ and $v = t + z$, then the metric would depend upon $u = t - z$, but not upon the combination $v = t + z$.

**Ans:**

$$X^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial x},$$

$$Y^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial y},$$

$$V^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial t} + \frac{\partial}{\partial z}.$$  

(b) Find $(x, y, z)$ as functions of $t$ for each of the mirrors.

**Ans:** Let $u^a$ be the four-velocity of one of the mirrors. With the Killing vectors, we know that $u^aX_a, u^aY_a$ and $u^aV_a$ are all constants of the motion. Also, $u^aX_a$ and $u^aY_a$ are zero for $t < z$, so they remain zero for all time, thus $u^x = u^y = 0$. From the normalization of $u^a$, we also know that

$$-1 = -(u^t)^2 + (u^z)^2 = - (u^t + u^z)(u^t - u^z).$$

But,

$$u^aV_a = -(u^t - u^z) = \text{constant},$$

and we conclude that $u^t + u^z$ as well as $u^t$ and $u^z$, individually, are also constants. Thus, $u^t, u^x, u^y$ and $u^z$ are all constant, and the coordinate positions of the particles $(x, y, z)$ remain at $(-\ell/2, 0, 0)$ and $(\ell/2, 0, 0)$ for all time.
(c) Find the distance between the mirrors as a function of $t$. If this question seems ill-posed, then assume that the time-scale over which $h$ changes is very much larger than $\ell$.

Ans:

$$D = \int_{-\ell/2}^{\ell/2} \sqrt{1+h} \, dx = \ell \sqrt{1+h} \approx \ell (1 + h/2)$$  \hspace{1cm} (1)

5. A space shuttle is orbiting a Schwarzschild black hole on a circular geodesic at radius $r$.

(a) What is the orbital frequency $\Omega$ of the shuttle as viewed by a distant observer, at rest with respect to the black hole?

Ans:

$$\Omega = \sqrt{m/r^3}$$

(b) A scientist on the shuttle shines a laser, of frequency $\nu$, out the front window in the $\phi$ direction. A second shuttle orbits the black hole on a geodesic at the same radius and in the same orbital plane but in the opposite direction. Just before the shuttles collide, what frequency does a scientist in the second shuttle measure for the laser light after it enters his front window?

Ans: The components of the four-velocity of the shuttle are

$$u^a = \left[ \frac{dt}{d\tau}, 0, 0, \frac{d\phi}{d\tau} \right] = \left[ \frac{E}{1-2m/r}, 0, 0, \frac{J}{r^2} \right]$$

$$= \frac{dt}{d\tau} \left[ 1, 0, 0, \frac{d\phi}{dt} \right] = (1 - 3m/r)^{-1/2} [1, 0, 0, \Omega],$$

with

$$E = (1 - 2m/r)(1 - 3m/r)^{-1/2} \quad \text{and} \quad \frac{dt}{d\tau} = (1 - 3m/r)^{-1/2}$$

for a circular geodesic.

The components of the four-velocity of the photon, when it is emitted, are

$$k^a = [k^t, 0, 0, k^\phi],$$

but $k^a$ is null so that

$$k^a k_a = -(1 - 2m/r)k^t \cdot r^2 k^\phi = 0 \Rightarrow k^\phi = k^t r^{-1} \sqrt{1 - 2m/r}. $$

Thus, we have

$$k^a = k^t [1, 0, 0, r^{-1} \sqrt{1 - 2m/r}].$$
Next, the frequency $\nu$ observed by the shuttle is

$$
\nu = -u^a k_a = \frac{dt}{d\tau} k^t \left[ (1 - 2m/r) - \Omega r \sqrt{1 - 2m/r} \right].
$$

The components of the four-velocity of the second shuttle are

$$
u^a_{ss} = \begin{bmatrix} \frac{dt}{d\tau}, 0, 0, \frac{d\phi}{d\tau} \end{bmatrix} = \begin{bmatrix} E, 0, 0, -\frac{J}{r^2} \end{bmatrix}$$

$$= \frac{dt}{d\tau} \begin{bmatrix} 1, 0, 0, \frac{d\phi}{dt} \end{bmatrix} = \frac{dt}{d\tau} [1, 0, 0, -\Omega],$$

And, the frequency $\nu_{ss}$ observed by the second shuttle is

$$
\nu_{ss} = -u^a_{ss} k_a = k^t \frac{dt}{d\tau} \left[ (1 - 2m/r) + \Omega r \sqrt{1 - 2m/r} \right].
$$

$$= \nu \frac{1 - 2m/r + \Omega r \sqrt{1 - 2m/r}}{1 - 2m/r - \Omega r \sqrt{1 - 2m/r}}. \quad (2)
$$

(c) Just before the collision of part (b), some of the laser light leaving the front window misses the oncoming second shuttle and ultimately goes through a distant observer’s laboratory. The distant observer is at rest with respect to the black hole. What frequency does the distant observer measure the light to have?

**Ans:** Now, $t^a \partial / \partial x^a = \partial / \partial t$ is a Killing vector field, so that $t^a k_a$ is a constant of the motion of the photon. The four-velocity of the observer at rest far from the black hole is also $t^a$, and the observed frequency at a large distance is $\nu_\infty = -t^a k_a$. Thus

$$
\nu_\infty = -t^a k_a = (1 - 2m/r) k^t
$$

$$= \nu \frac{1 - 2m/r}{(dt/d\tau) \left[ 1 - 2m/r - \Omega r \sqrt{1 - 2m/r} \right]}
$$

$$= \nu \frac{\sqrt{1 - 3m/r}}{\left[ 1 - \Omega r \sqrt{1 - 2m/r} \right]}.
$$

A loose interpretation of this result has the numerator representing the gravitational redshift, and the denominator representing a Doppler blue shift.