Gravitational self-force effects on orbits around a non-rotating black hole

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- The Lorenz gauge versus the “true harmonic” gauge.
- Gauge invariant quantities for describing “circular” geodesics in $g_{ab}^{\text{Schw}} + h_{ab}$.
- Actual results for gravitational self-force effects on circular orbits in the Schwarzschild geometry.
- A scheme for doing second order metric perturbations with a point mass (small, black hole) source.

det, Capra VIII, Rutherford Appleton Laboratory, Oxford, 2005
The true harmonic gauge versus the Lorenz gauge

With the Schwarzschild geometry there are four different scalar fields

\[ T = t, \quad X = (r_{\text{Schw}} - m) \sin \theta \cos \phi, \quad Y = (r_{\text{Schw}} - m) \sin \theta \sin \phi, \quad Z = (r_{\text{Schw}} - m) \cos \theta, \]

for which

\[ \nabla^a \nabla_a T = \nabla^a \nabla_a X = \nabla^a \nabla_a Y = \nabla^a \nabla_a Z = 0. \]

These four scalar fields may be used for harmonic coordinates, but for the time being we continue to use Schwarzschild coordinates. We consider an arbitrary perturbation of the metric and seek a gauge transformation which results in these same scalar fields being harmonic functions in the perturbed metric \( g_{ab} + h_{ab} \).

This is surprisingly simple to do. The condition that a scalar field \( X \) be harmonic in \( g_{ab} + h_{ab} \) is that \( \nabla^a \nabla_a (g + h) X = 0 \). Three other, similar equations for \( T, Y \) and \( Z \) should also hold. Together, these give us the four gauge conditions for the true-harmonic gauge.
With no details being given, we define \( \bar{h}_{ab} = h_{ab} - \frac{1}{2} g_{ab} h \), where \( h = g^{ab} h_{ab} \).

The true-harmonic gauge condition for the perturbed geometry is then

\[
\nabla_a (\bar{h}^{ab} \nabla_b X) = 0.
\]

Note that the gauge condition is covariant and distinct from what is usually described as the Lorenz gauge condition, \( \nabla_a \bar{h}^{ab} = 0 \).

Under a gauge transformation,

\[
\bar{h}_{ab}^{\text{new}} = \bar{h}_{ab}^{\text{old}} - \nabla^a \xi^b - \nabla^b \xi^a + g^{ab} \nabla^c \xi_c.
\]

Thus, with an initial metric perturbation \( h_{ab}^{\text{old}} \), the gauge vector \( \xi^a \) to transform to the true-harmonic gauge must satisfy

\[
\nabla_a (\bar{h}_{ab}^{\text{old}} \nabla_b X) = \left( \nabla_a \nabla^a \xi^b + R^b_c \xi_c \right) \nabla_b X + 2 (\nabla^a \xi^b) \nabla_a \nabla_b X,
\]

as well as three other similar equations for \( T, Y, \) and \( Z \).
Gauge Invariance

“The perturbation in some quantity is the difference between the value it has at a point in the physical (perturbed) spacetime and the value at the corresponding point in the background spacetime. A gauge transformation induces a coordinate transformation in the physical spacetime, but it also changes the point in the background spacetime corresponding to a given point in the physical spacetime. Thus, even if a quantity is a scalar under coordinate transformations, the value of the perturbation in the quantity will not be invariant under gauge transformations if the quantity is nonzero and position dependent in the background spacetime.” (Bardeen 1980)

A gauge transformation is a small change in coordinates, \(x^a_{\text{new}} = x^a + \xi^a\), with \(\xi^a = O(\mu)\) which changes the metric perturbation, \(h^\text{new}_{ab} = h_{ab} - 2\nabla^a(\xi^b) + O(\mu^2)\). But, for an arbitrary \(\xi^a\), the tensor \(2\nabla^a(\xi^b)\) is explicitly a homogeneous solution of the perturbed Einstein equations, and the perturbed Einstein tensor is therefore invariant under such a gauge transformation.
Gauge invariant quantities for “circular” geodesics in $g_{ab}^{\text{Schw}} + h_{ab}^R$

Self-force analysis implies that a point mass $\mu$ moves along a geodesic of the perturbed metric $g_{ab}^{\text{Schw}} + h_{ab}$, where $h_{ab} \equiv h_{ab}^R$ is $C^1$. This geodesic equation is

$$\frac{du_a}{ds} = \frac{1}{2} u^b u^c \frac{\partial}{\partial x^a} (g_{bc} + h_{bc})$$

Let $R(s)$ be $r$ for the particle, and define

$$u_t = -E, \quad u_\phi = J \quad \text{and} \quad u^r = \dot{R},$$

$$u^a = \left( \frac{E + u^b h_{tb}}{1 - 2M/r}, \dot{R}, 0, \frac{J - u^b h_{\phi b}}{r^2} \right).$$

$E$ and $J$ are similar to the particle’s energy and angular momentum per unit rest mass.
The components of the geodesic equation

\[
\begin{align*}
\frac{dE}{ds} &= -\frac{1}{2} u^a u^b \frac{\partial h_{ab}}{\partial t} \\
\frac{dJ}{ds} &= \frac{1}{2} u^a u^b \frac{\partial h_{ab}}{\partial \phi} \\
\frac{d}{ds} \left( \frac{r \dot{R}}{r - 2M} + u^a h_{ar} \right) &= \frac{1}{2} u^a u^b \frac{\partial}{\partial r} (g_{ab} + h_{ab})
\end{align*}
\]

Assume that \( \ddot{R} = O(h^2) \) and that \( \dot{R} = O(h) \) — this is consistent with quasi-circular evolution.

The normalization of \( u^a \) is a first integral of the geodesic equation,

\[
1 = \frac{E^2}{1 - 2M/r} - \frac{J^2}{r^2} + u^a u^b h_{ab}.
\]
Symmetries for “circular” orbits

Neither $\partial / \partial t$ nor $\partial / \partial \phi$ is a Killing vector of $g_{ab} + h_{ab}$, but the combination, $k^a \frac{\partial}{\partial x^a} = \partial / \partial t + \Omega \partial / \partial \phi$ is a Killing vector, $\mathcal{L}_k h_{ab} = 0$, and $u^a$ is tangent to a trajectory of $k^a$. Thus, at a “circular” orbit $u^a \partial_a h_{bc} = 0$ in Schwarzschild coordinates.

The “circular” orbits of the perturbed geometry are obtained from the $r$-component of the geodesic equation, and the normalization condition, and the facts that

$$\dot{E} \sim O(h) \quad \dot{J} \sim O(h)$$

A gauge transformation

Only a gauge transformation in the radial coordinate $r_{\text{new}} = r_{\text{old}} + \xi r$ induces a change in

$$\Delta(u^a u^b \partial_r h_{ab})_\mu = -\frac{6M}{r^2(r-3M)} \xi r$$

evaluated at the particle. Also, $u^a u^b h_{ab}|_\mu$ is invariant under any gauge transformation. These facts imply that the quantities below are gauge invariant.
Gauge invariant quantities

Consequences of the geodesic equation are

\[ (u^t)^2 = \left( \frac{dT}{ds} \right)^2 = \frac{(E + u^b h_{tb})^2}{(1 - 2M/r)^2} \]
\[ = \frac{r}{r - 3M} \left( 1 + u^a u^b h_{ab} - \frac{r}{2} u^a u^b \partial_r h_{ab} \right) \]

\[ (u^\phi)^2 = \left( \frac{d\Phi}{ds} \right)^2 = \frac{1}{r^4} (J - u^b h_{\phi b})^2 \]
\[ = \frac{r - 2M}{r(r - 3M)} \left[ \frac{M(1 + u^a u^b h_{ab})}{r(r - 2M)} - \frac{1}{2} r u^a u^b \partial_r h_{ab} \right] \]

\[ \Omega^2 = \left( \frac{u^\phi}{u^t} \right)^2 = \frac{M}{r^3} - \frac{r - 3M}{2r^2} u^a u^b \partial_r h_{ab} \]

\[ \frac{dE}{dt} = -\frac{1}{2} \sqrt{1 - 3M/r} \ u^a u^b \partial_t h_{ab} \quad \frac{dJ}{dt} = \frac{1}{2} \sqrt{1 - 3M/r} \ u^a u^b \partial_\phi h_{ab} \]
Consider

\[
(E - \Omega J) \frac{dT}{ds} = E \frac{dT}{ds} - J \frac{d\Phi}{ds}
\]

\[
= \frac{E(E + u^b h_{tb})}{1 - 2M/r} - \frac{J}{r^2} (J - u^b h_{\phi b})
\]

\[
= \frac{E^2}{1 - 2M/r} - \frac{J}{r^2} + \frac{Eu^b h_{tb}}{1 - 2M/r} + \frac{J u^b h_{\phi b}}{r^2}
\]

\[
= 1 - u^a u^b h_{ab} + u^a u^b h_{ab} = 1
\]

Therefore

\[
k^a u_a = E - \Omega J = (dT/ds)^{-1} = (u^t)^{-1}
\]

is also gauge invariant.
**Physical interpretation of the gauge invariant $u^t$**

Let a light source be near the small mass $\mu$. Let the tangent vector to an affinely parameterized null geodesic of a photon from this light source be $\nu^a$. The energy $\mathcal{E}_{em}$ of the photon, as emitted near $\mu$, is proportional to $u^a\nu_a$, so the ratio of the energies as measured by an observer and as emitted is

$$\frac{\mathcal{E}_{ob}}{\mathcal{E}_{em}} = \frac{u^a\nu_a|_{ob}}{u^a\nu_a|_{em}}$$

With $k^a$ a Killing vector field, $k^a\nu_a$ is constant along the path of the photon. At emission, $u^a_{em} \propto k^a$ so that $u^a_{em} = u^t k^a|_{em}$. Let the photon be observed at a large distance away from the black hole along the $z$-axis.
It follows that

\[
\frac{E_{ob}}{E_{em}} = \frac{u^a v_a|_{ob}}{u^a v_a|_{em}}, \quad \text{with } u^t_\infty = 1 \text{ this becomes}
\]

\[
= \frac{v_t^\infty}{u^t(k^a v_a)_em} = \frac{v_t^\infty}{u^t(k^a v_a)_\infty}, \quad \text{because } k^a u_a = \text{constant along the geodesic},
\]

\[
= \frac{v_t^\infty}{u^t(v_t^\infty + \Omega v_\phi^\infty)} = \frac{1}{u^t} - \frac{\Omega v_\phi^\infty}{u^t(v_t^\infty + \Omega v_\phi^\infty)}
\]

\[
= \frac{1}{u^t}, \quad \text{because } v_\phi^\infty = 0 \text{ at a large distance along the } z\text{-axis.}
\]

Thus, the gauge invariant \( u^t = 1/(E - \Omega J) \) gives the redshift of a photon, emitted from \( \mu \), when the photon is observed on the \( z\)-axis at a large distance.
A surprising (to me) fact

• An arbitrary metric perturbation of a spherically symmetric background spacetime retains some residual spherical symmetry:

• The perturbed spacetime may be foliated by a *unique* family \( \{\Sigma\} \) of two-spheres, which are individually spherically symmetric, even while the spacetime as a whole is not. There is a gauge with \( h_{\theta\theta} = h_{\theta\phi} = h_{\phi\phi} = 0 \).

• The existence of the two-spheres in the perturbed geometry, permits the geometrical definition of a scalar field \( R \) from the area of each \( \Sigma \).

• The gauge where \( r_{\text{schw}} = R \) is called the *Easy Gauge*. The metric perturbations in the EZ gauge may be interpreted as being “gauge invariant” in the same manner that Moncrief showed that the Regge-Wheeler-gauge metric perturbations could be described as being “gauge invariant.” The relationship between the EZ gauge variables and the Regge-Wheeler-Moncrief variables is not just algebraic, but also involves differentiation. As a result, the form of the perturbed Einstein equations in the EZ gauge differs from that in the Regge-Wheeler gauge. There are geometrical interpretations of all of the EZ gauge-invariant scalars in terms of the lapse and shift using a foliation of the 3-geometry in terms of \( \{\Sigma\} \).
Definitions of two “radial R” quantities

The orbital frequency of a circular, Newtonian binary of masses $M$ and $\mu$ is

$$\Omega^2 = \frac{M + \mu}{r^3}$$

where $r$ is the separation between $M$ and $\mu$. For a general-relativistic, extreme mass ratio binary we define $R\Omega$ by

$$\Omega^2 = \frac{M + \mu}{R_\Omega^3}$$ defines $R\Omega$.

In a circular Newtonian binary, the radius of the orbit of $\mu$ is

$$\text{distance to center of mass} = \text{separation}/(1 + \mu/M).$$

In the extreme mass ratio limit, this becomes

$$\text{distance to center of mass} = R\Omega(1 - \mu/M).$$
An actual well-defined consequence of the gravitational self-force

- The redshift of a photon from $\mu$ is

$$\frac{\mathcal{E}_{\text{ob}}}{\mathcal{E}_{\text{em}}} = \frac{1}{u^t}$$

when observed at a large distance along the $z$-axis.
- I have not yet calculated this redshift in the post-Newtonian approximation.
A second well-defined consequence of the gravitational self-force

\[ \frac{(R_{EZ} - R_{\Omega})}{R_{\Omega}} \times \frac{M}{\mu} \]

- At large \( R_{\Omega} \), the areal radius of the geometrical two-spheres

\[ R_{EZ} \approx R_{\Omega}(1 - \mu/M) = \text{Newtonian distance to the center of mass} \]

- \( R_{EZ} \) does not (yet?) appear to be physically observable.
- I have not yet calculated \( R_{EZ} \) in the post-Newtonian approximation.
Second order perturbation theory with a point mass, schematically

Define the parts of the Einstein tensor of various orders in $h$ by

$$G(g + h) = G^{(1)}(g, h) + G^{(2)}(g, h) + G^{(3)}(g, h) + \ldots$$

$G^{(1)}(g, h)$ looks like a wave operator on $h$; $G^{(2)}(g, h)$ looks like "$\nabla h \nabla h$" or "$h \nabla \nabla h$".

At second order solve

$$G^{(1)}(g, h) + G^{(2)}(g, h) = 8\pi T \quad \text{or} \quad G^{(1)}(g, h^R + h^S) + G^{(2)}(g, h^R + h^S) = 8\pi T$$

by using

$$G^{(1)}(g, h^R) = -G^{(2)}(g, h^R) - [G^{(1)}(g + h^R, h^S) - G^{(2)}(g, h^S) + 8\pi T]$$

If we know $h^S$ well enough then

$$[G^{(1)}(g + h^R, h^S) - G^{(2)}(g, h^S) + 8\pi T] = O(\mu r/\mathbb{R}^4) = C^0$$

- The numerical solution for $h^R$ will be $C^2$
- Except for being continuous but non-differentiable at the point mass, the source is relatively smooth.
\[ g \sim \eta \quad \mu/r \quad \mu^2/r^2 \quad \mu^3/r^3 \quad \cdots \]
\[ \mu/\mathcal{R} \quad \mu^2/\mathcal{R}^2 \quad \mu^3/\mathcal{R}^3 \quad \cdots \]
\[ \mu r/\mathcal{R}^2 \quad \mu^2 r/\mathcal{R}^3 \quad \mu^3 r/\mathcal{R}^4 \quad \cdots \]
\[ \mu^2 r/\mathcal{R}^3 \quad \mu^3 r^2/\mathcal{R}^4 \quad \cdots \]
\[ \mu^3 r^2/\mathcal{R}^4 \quad \cdots \]
\[ g^{\text{Schw}} \quad 0 \quad 2h' \quad 3h' \quad 4h' \quad \cdots \]

\[ = g^0 + h_R \quad = h^\mu \]
\[ = h^\mu_2 \quad = h^\mu_3 \]

(1)

- The point mass moves along a geodesic of \( g + h^R \)
- At higher order, as long as the motion is geodesic in \( g + h^R \), the formulation of higher order perturbation theory is relatively straightforward.
Conclusions

- Other conservative self-force effects will be studied for slightly non-circular orbits of Schwarzschild. These include the self-force effects on the precession of the perihelion of an orbit, and on the orbital frequency of the innermost stable circular orbit.

- There seems to be no fundamental difficulty in doing second order perturbation theory.

- Second order calculations will certainly be able to provide improved wave-forms for LISA and also be able to test convergence of the post-Newtonian approximation.