Problem 2

In the figure on the board a metal rod is forced to move with constant velocity 60 cm/s along two parallel metal rails connected with a strip of metal at one end. A magnetic field of magnitude $B = 0.2$ T points out of the board. The rails are separated by $L = 45$ cm, and the rod has resistance $50 \, \Omega$ (a) What is the voltage induced on the metal rod? (b) Give the direction and magnitude of the induced current in the metal rod (c) What’s the force on the rod? Give both magnitude and direction (d) At what rate is energy being transferred to thermal energy?

Solution

(a) The induced voltage $\xi$ is given by Faraday’s law:

$$\xi = \oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

where the closed path in the integration on the left-hand side can be taken around the closed loop formed by the rod and the rail. If we call $x$ the distance between the rod and the right most side of the rail, we have that $x = x_0 + vt$, where $x_0$ is the initial position of the rod at $t = 0$ (which will be irrelevant at the end). Defining $d\vec{A}$ to point out of the page\(^1\), and since the magnetic field is uniform, the integral over $d\vec{A}$ is trivial and simply equal to $|\vec{B}| A$. Therefore, the right-hand side is

$$-\frac{d}{dt} \int \vec{B} \cdot d\vec{A} = -\frac{d}{dt}(|\vec{B}|xL) = |\vec{B}|Lv$$

Thus,

$$|\xi| = |\vec{B}|Lv = 0.2 \cdot 0.45 \cdot 0.6 = 0.54 \text{ Volts}$$

\(^1\)If you choose the opposite direction you will simply change the sign of $\xi$. This does not bother us here since we’re only interested in the magnitude of the voltage created on the rod.
(b) The magnitude of the current is given by Ohm’s law, i.e. $i = \xi/R = 0.54/50 = 0.0011$ A or 1.1 mA. Its direction can be obtained by looking at the direction of the force acting on each charge free to move inside the rod according to $\vec{F} = q \vec{v} \times \vec{B}$. Given the directions of $\vec{v}$ and $\vec{B}$, and the fact that the direction of the current is defined as the direction in which positive charges would flow, we have that $\vec{F}$ points up. Therefore, the currents goes up the rod. This will create a full current going in the clockwise direction along the rod-rail closed loop.

(c) Now that we have positive charges moving in the up direction inside the rod, they immediately feel the presence of the magnetic field as soon as the move up. Thus, again we need to use $\vec{F} = q \vec{v} \times \vec{B}$ to find the direction of the new force acting on them. Now each charge also has a vertical component in its velocity, thus this will produce a force going to the right. Thus, as a result, the rod responds to the external pull to the left with a force acting to the right as if it was trying to resist the pull\(^2\). The magnitude of the force is given by $|F| = iLB = 0.0011 \cdot 0.45 \cdot 0.2 = 9.9 \times 10^{-5}$ N.

(d) Remember that rate of energy transfer is what we call power. As you might recall, the resistor dissipates energy in terms of heat and the power dissipated in a resistor is given by $P = i^2R$. Thus, the rate at which energy is being transfered to thermal energy is simply $P = 0.0011^2 \cdot 50 = 6.05 \times 10^{-5}$ Watts.

**Problem 3**

A circular region in a $xy$ plane is penetrated by a uniform magnetic in the positive direction of the $z$ axis (*Note: The magnetic field exists only inside the loop. $\vec{B}$ is zero outside*). The magnitude of $\vec{B}$ (in teslas) increases with time $t$ (in seconds) according to $B = at$, where $a$ is a constant. The magnitude $E$ of the electric field set up by that increase in the magnetic field is given by the figure where $E$ is plotted versus the distance $r$ from the center of the loop. Find $a$.

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\(^2\)You can also think of this as a consequence of energy conservation.
Solution

Let's compute the magnetic field created by this circular region both inside and outside the circle. Let's call \( R \) the radius of this circle even though we don't know a priori what is its value.

Now focus in a circular region with radius \( r < R \). The voltage \( \xi \) induced along the circumference of radius \( r \) is given by

\[
\xi = \oint \vec{E} \cdot d\vec{l}
\]

(4)

**Note:** We used to think that, since this is a closed contour integral of the electric field along the loop, it should be zero. That was true before where there were no time varying fields. Now, with fields that vary with time, that result is no longer true and the expression above can be a nonzero number which is fact \( -\frac{d\Phi_B}{dt} \).

Due to Faraday's Law, this induced voltage has a magnitude given by

\[
\xi = \frac{d}{dt} \int \vec{B} \cdot d\vec{A}
\]

(5)

therefore

\[
\oint \vec{E} \cdot d\vec{l} = \frac{d}{dt} \int \vec{B} \cdot d\vec{A}
\]

\[
E2\pi r = \frac{d}{dt} \left( a\pi r^2 \right)
\]

\[
2E = ar
\]

Thus

\[
E = \frac{ar}{2}
\]

(6)

Note that the electric field increases linearly with the distance with respect to the center of the circle, agreeing with the straight line behavior we see in the left part of the graph.

Now let's derive the electric field for the region outside the circle, i.e. for \( r > R \).

\[
\oint \vec{E} \cdot d\vec{l} = \frac{d}{dt} \int \vec{B} \cdot d\vec{A}
\]

\[
E2\pi r = \frac{d}{dt} \left( a\pi R^2 \right)
\]

\[
2Er = aR^2
\]

Therefore

\[
E = \frac{aR^2}{2r}
\]

(7)

Note that now the electric field decreases with the distance as \( 1/r \), perfectly matching the decreasing behavior we see in the right part of the graph, right after the kink located at
$r = r_s/2$. Since the electric field is a continuous function of the position, i.e. the radius, by gluing the two solutions at the discontinuity point $r = r_s/2$ we realize that $R = r_s/2$.

Hence

$$E(r) = \begin{cases} \frac{ar}{2} & \text{if } r > r_s/2 \\ \frac{aR^2}{2r} & \text{if } r < r_s/2 \end{cases}$$

At $r = r_s/2$ we have

$$E(r = r_s/2) = E_{max} = \frac{aR^2}{2(r_s/2)} = \frac{aR^2}{r_s}$$

from where we obtain

$$a = \frac{r_s E_{max}}{R^2}$$