1. Calculate the Joule-Thompson coefficient \( \left( \frac{\partial u}{\partial v} \right)_T \), where \( u \) is the internal energy and \( v \) is the volume, for a gas with equation of state \( p = \frac{RT}{(v-b) - \frac{a}{v^2}} \).

[Hint: use \( du = Tds - pdv \) and Maxwell relation \( \left( \frac{\partial s}{\partial v} \right)_T = \left( \frac{\partial p}{\partial T} \right)_V \).]

Start with \( du = Tds - pdv \) and \( p = \frac{RT}{(v-b) - \frac{a}{v^2}} \). Consider \( s = s(v,T) \).

Then

\[
du = T \left[ \left( \frac{\partial s}{\partial v} \right)_T \ dv + \left( \frac{\partial s}{\partial T} \right)_v \ dT \right] - pdv. \tag{1}
\]

Now we see that the derivative requested is

\[
\left( \frac{\partial u}{\partial v} \right)_T = T \left( \frac{\partial s}{\partial v} \right)_T - p = T \left( \frac{\partial p}{\partial T} \right)_v - p = T \frac{R}{v-b} - p = \frac{a}{v^2} \tag{2}
\]

2. Consider the triangle in the \((x, y)\) plane with vertices at \((-1,0)\), \((1,0)\), and \((0,1)\).

Evaluate the closed line integral

\[
I = \oint (-y\hat{x} + x\hat{y}) \cdot d\vec{r} \tag{3}
\]

around the boundary of the triangle in the anticlockwise direction.

\[
d\vec{r} = \hat{x}dx + \hat{y}dy, \text{ so } (-y\hat{x} + x\hat{y}) \cdot d\vec{r} = -ydx + xdy.
\]

On leg \((-1,0) \rightarrow (1,0)\) we have \( y = 0 \), so integral is \( \int_{-1}^{1}(-y)dx = 0 \). On the leg \((1,0) \rightarrow (0,1)\) we have \( y = -x + 1 \), so integral is \(-\int_{1}^{0}(-x+1)dx + \int_{0}^{1}(1-y)dy = \frac{1}{2} + \frac{1}{2} = 1 \). On the path \((0,1) \rightarrow (-1,0)\) we have \( y = x + 1 \), so integral is \(-\int_{0}^{-1}(x+1)dx + \int_{1}^{0}(y-1)dy = \frac{1}{2} + \frac{1}{2} = 1 \). So total line integral is 2.

3. Consider the parabola \( y = 4 + 5x^2 \). Find the closest point to the origin on this curve by the method of Lagrange multipliers.

I actually did it 3 ways to illustrate the possibilities:

(a) substituting explicitly into the distance formula for \( y(x) \) and solve the conventional 1D minimization problem.
(b) substituting explicitly into the distance formula for \( x(y) \) and solve the conventional 1D minimization problem.
using the method of Lagrange multipliers.
We’ll minimize \( x^2 + y^2 \) rather than \( \sqrt{x^2 + y^2} \) as usual:

(a) \( y = 4 + 5x^2 \) so \( x^2 + y^2 = x^2 + (4 + 5x^2)^2 \). Minimize

\[
\frac{d}{dx} \left( 25x^4 + 41x^2 + 16 \right) = 100x^3 + 82x = 0 \quad \Rightarrow \quad x = 0, \ y = 4 \quad \checkmark \quad (4)
\]

(b) \( x^2 = (y - 4)/5 \), so minimize \((y - 4)/5 + y^2\):

\[
\frac{d}{dy} \left( \frac{y - 4}{5} + y^2 \right) = \frac{1}{5} + 2y = 0 \quad \Rightarrow \quad y = -\frac{1}{10}, \quad (5)
\]

But this value cannot lie on the parabola, so it must be spurious somehow. Going back, we see that at this value of \( y \) the quantity \( x^2 \) on the parabola becomes negative, so this is not a valid solution for a point \( x, y \) on the parabola. The minimum must take place on the boundary of the set of \( x, y \) lying on the parabola, i.e. \( y = 4 \), implying \( x = 0 \).

(c) Take \( f = x^2 + y^2 \), function to be minimized in unconstrained space according to M. Lagrange is

\[
F = f + \lambda (4 + 5x^2 - y) \quad (6)
\]

So 3 equations for a minimum are

\[
\frac{\partial F}{\partial x} = 0 = 2x + 10x\lambda ; \quad \frac{\partial F}{\partial y} = 0 = 2y - \lambda ; \quad \frac{\partial F}{\partial \lambda} = 0 = 4 + 5x^2 - y. \quad (7)
\]

1st equation admits a solution \( x = 0 \) or \( \lambda = -1/5 \). The first one is correct, yields \( y = 4 \) from constraint (3rd) equation. Second one gives \( y = -1/10 \) again, this is the spurious solution discussed above.

4. Calculate the total derivative \( dy/dx \) for \( x = \frac{y-2}{y+4} \) in two ways:

(a) (4 pts.) explicitly solve for \( y(x) \)

(b) (4 pts.) use implicit differentiation.

(c) (2pts.) Verify that your answer is the same in both cases.

(a) Solve by finding \( y(x) \), \( y = (4x + 2)/(1 - x) \), so \( dy/dx = 6/(1 - x)^2 \).

(b) Implicitly:

\[
x(y + 4) = y - 2 \quad \Rightarrow \quad dx(y + 4) + xdy = dy
\]

\[
\Rightarrow dy = \frac{dx(y + 4)}{1 - x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y + 4}{1 - x} \quad (8)
\]

(c)

\[
\frac{4 + y}{1 - x} = \frac{4 + \frac{4x + 2}{1 - x}}{1 - x} = \frac{6}{(1 - x)^2} \quad (9)
\]
5. Using the properties of the Levi-Civita symbol, verify the vector identity

\[ \vec{A} \times (\vec{\nabla} \times \vec{A}) = \frac{1}{2} \vec{\nabla}(A^2) - (\vec{A} \cdot \vec{\nabla})\vec{A} \tag{10} \]

Repeated summation index convention:

\[ \left( \vec{A} \times (\vec{\nabla} \times \vec{A}) \right)_i = \epsilon_{ijk}A_j(\vec{\nabla} \times \vec{A})_k = \epsilon_{i j k}A_j\epsilon_{k \ell m} \nabla_\ell A_m = (\epsilon_{i j k} \epsilon_{\ell m k})A_j \nabla_\ell A_m \]

\[ = (\delta_{i \ell} \delta_{jm} - \delta_{im} \delta_{j \ell})A_j \nabla_\ell A_m = \frac{1}{2} \nabla_i A^2 - (\vec{A} \cdot \vec{\nabla})A_i \tag{11} \]

Note that I used \( A_j \nabla_i A_j = \frac{1}{2} \nabla_i A^2 \), just the product rule.

6. (Extra credit, 5 pts.) In the integral

\[ I = \int_{x=0}^{1/2} \int_{y=x}^{1-x} \left( \frac{x - y}{x + y} \right)^2 dy dx, \tag{12} \]

make the transformation

\[ x = \frac{1}{2}(r - s) ; \quad y = \frac{1}{2}(r + s), \tag{13} \]

and evaluate I. [Hint: sketch the area of integration in \( x - y \) plane, then draw the \( r \) and \( s \) axes. Determine the area of \( r - s \) integration.]

FIG. 1: Variables \( x, y \) transformation to \( r, s \).
Jacobian of inverse transformation \( r = x + y, \ s = y - x \) is

\[
J \left( \begin{array}{c} x, y \\ r, s \end{array} \right) = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2},
\]

so

\[
I = \int_0^1 dr \int_0^r ds \left( \frac{s}{r} \right)^2 \cdot \frac{1}{2} = \int_0^1 \frac{1}{r^2} dr \left( \frac{1}{2} \frac{s^3}{3} \right) = \int_0^1 \frac{r}{6} dr = \frac{1}{12}.
\]

7. (Extra credit, 5 pts.) Planck's theory of quantized oscillators led to an average energy

\[
\langle \epsilon \rangle = \frac{\sum_{n=1}^{\infty} n \epsilon_0 \exp(-n\epsilon_0/kT)}{\sum_{n=0}^{\infty} \exp(-n\epsilon_0/kT)},
\]

where \( \epsilon_0 \) was a constant energy. Find \( d\langle \epsilon \rangle/dT \) in closed form (evaluate all sums).

First call \( \alpha = \epsilon_0/kT \). Then note that \( \sum_n \exp(-\alpha n) = (1 - \exp(-\alpha))^{-1} \) is the sum of the geometric series, and that

\[
\frac{d}{d\alpha} \sum_n \exp(-\alpha n) = -\sum_n n \exp(-\alpha n)
\]

and

\[
\frac{d}{d\alpha} \sum_n \exp(-\alpha n) = \frac{d}{d\alpha} \left( \frac{1}{1 - e^{-\alpha}} \right) = -\frac{e^{-\alpha}}{(1 - e^{-\alpha})^2}
\]

so

\[
\langle \epsilon \rangle = -\epsilon_0 \frac{e^{-\alpha}}{1 - e^{-\alpha}} = \frac{\epsilon_0}{e^\alpha - 1}.
\]

So

\[
\frac{d\langle \epsilon \rangle}{dT} = \frac{d\langle \epsilon \rangle}{d\alpha} \frac{d\alpha}{dT} = \epsilon_0 \cdot \frac{\epsilon_0}{k} \cdot \frac{-1}{T^2} \frac{e^\alpha}{(e^\alpha - 1)^2} = k \alpha^2 \frac{e^\alpha}{(e^\alpha - 1)^2}.
\]