The catenary curve (from the Latin for “chain”) is the shape assumed by a uniform chain of fixed length supported at its ends under the influence of gravity. Let the curve be described by a function \( y(x) \) with endpoints \( y(x_1) = y_1, y(x_2) = y_2 \). Let \( \mu \) be the (constant) mass per unit length. The shape is then found by minimizing the gravitational potential energy,

\[
E[y(x)] = \sum mgh = \int \mu g y(x) \, ds \int \mu g y \sqrt{1 + y'^2} \, dx,
\]

while fixing the length

\[
s[y(x)] = \int_{x_1}^{x_2} \sqrt{1 + y'^2} \, dx = \ell.
\]

(A functional \( J[y(x)] \) is an operation that takes as its argument the function \( y(x) \) and yields a number: energy, length, “action,” . . . ) Impose the constraint with a Lagrange multiplier and extremize the action

\[
J = E + \lambda (s - \ell) = \int (\lambda + \mu g y) \sqrt{1 + y'^2} \, dx - \lambda \ell.
\]

As in Section 6.5 in the text, varying \( y(x) \) leads to the Euler equation,

\[
\frac{\partial f}{\partial y} - \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = \mu g \sqrt{1 + y'^2} - \frac{d}{dx} \left[ \frac{(\lambda + \mu g) y'}{\sqrt{1 + y'^2}} \right]
\]

\[
= \frac{\mu g (1 + y'^2)^2}{(1 + y'^2)^3} - \frac{\lambda y'' + \mu g (y y'' + y'^2 + y'^4)}{(1 + y'^2)^3} = \frac{\mu g (1 + y'^2 - y y'') - \lambda y''}{(1 + y'^2)^3} = 0.
\]

The expression is simpler than it might have been because the derivative

\[
\frac{d}{dx} \frac{y'}{\sqrt{1 + y'^2}} = \frac{y''}{(1 + y'^2)^3}
\]

comes out nice (for exponents other than 2 inside the square root there are more terms), and because \( y'^4 \) terms cancel between the \( \partial f/\partial y \) and \( \partial f/\partial y' \) terms. Variation of \( \lambda \) leads to

\[
\frac{\partial J}{\partial \lambda} = \int_{x_1}^{x_2} \sqrt{1 + y'^2} \, dx - \ell = 0.
\]

Thus, after all this effort we are led to a deceptively simple-looking differential equation plus an integral constraint,

\[
(\lambda/\mu g + y) y'' = 1 + y'^2, \quad \int_{x_1}^{x_2} \sqrt{1 + y'^2} \, dx = \ell.
\]

which we must solve for given boundary conditions.
The solution makes use of properties of the hyperbolic functions

\[ \cosh x = \frac{1}{2}(e^x + e^{-x}), \quad \sinh x = \frac{1}{2}(e^x - e^{-x}), \]

and in particular the properties

\[ \frac{d}{dx} \cosh x = \sinh x, \quad \frac{d}{dx} \sinh x = \cosh x, \quad \cosh^2 x = 1 + \sinh^2 x. \]

Thus, the function \( y = \cosh x \) satisfies \( yy'' = 1 + y'^2 \); This is not yet the solution, because we still have to account for boundary conditions and the constraint. First note that \( yy'' = 1 + y'^2 \) remains true after a scaling, \( y = a \cosh(x/a) \). This is good because \( x \) actually has units, while the argument of \( \cosh \) must be dimensionless; and also because it will allow us to satisfy the length constraint. We can also translate the minimum anywhere we need by shifting \( x \) to \( x - b \). Finally, we must address the \( \lambda/\mu g \) term, but that can be done by adding a constant to \( y \). So, with all of this we have the general shape of the curve

\[ y(x) = a \cosh\left(\frac{x-b}{a}\right) + c, \quad y' = \sinh\left(\frac{x-b}{a}\right), \quad y'' = \frac{1}{a} \cosh\left(\frac{x-b}{a}\right). \]

This function satisfies the differential equation for any values of \( a, b, \) and \( c \), as long as \( \lambda/\mu g + c = 0 \), which determines \( \lambda \). The value of the coefficients \( a, b, \) and \( c \) are determined by the length constraint,

\[ s = \int_{x_1}^{x_2} \sqrt{1 + y'^2} \, dx = \int_{x_1}^{x_2} \sqrt{1 + \sinh^2\left(\frac{x-b}{a}\right)} \, dx \]
\[ = \int_{x_1}^{x_2} \cosh\left(\frac{x-b}{a}\right) \, dx = a \left[ \sinh\left(\frac{x_2-b}{a}\right) - \sinh\left(\frac{x_1-b}{a}\right) \right] = \ell. \]

plus boundary conditions.

As an example, let \( x_1 = -\frac{1}{2}d \) and \( x_2 = \frac{1}{2}d \), with \( y_1 = y_2 = h \). Symmetry in \( \pm x \) says \( b = 0 \). Then the length constraint says

\[ \frac{\sinh(d/2a)}{(d/2a)} = \frac{\ell}{d}, \]

which, since \( \sinh x > x \), has a solution for any \( \ell \geq d \) and serves to determine \( a \); and the value \( y = h \) at \( x = \pm d \) fixes \( c \).

The figure (following page) shows results for chains of various lengths. The longest one falls below \( y = 0 \).
Catenary Curves