Here are some problems that probably won’t appear on this year’s exam. First, a 2-d radiation problem:

1. A long, straight wire carries a current that is driven (somehow) to vary as \( I = I_0 \cos \omega t \).

   (a) What are the directions of the electric and magnetic fields? (Explain your reasoning.)

Use symmetry: the problem is invariant under translations along the wire axis (let this be the \( z \)-axis) and under rotations about the axis. A radial magnetic field is not allowed, as it would give a nonzero magnetic flux through a cylindrical surface; and the magnetic field is perpendicular to currents, and so there is no \( z \)-component. A radial electric field by Gauss’s law for a cylindrical surface would have to enclose a charge, and an azimuthal field has a circulation, which would require a time-varying \( B_z \). So: \( E \) is in the \( z \)-direction, \( B \) is in the \( \phi \)-direction.

Or, use analytic arguments: the vector potential \( A \) must be in the \( z \)-direction, and from symmetry can depend only on \( \rho \), \( A = A(\rho) \hat{z} \). Such a vector potential has no divergence, and so \( \Phi = 0 \). So,

\[
\mathbf{B} = \nabla \times \mathbf{A} = \frac{\partial A}{\partial \rho} \hat{\rho} \times \hat{z} = -\frac{\partial A}{\partial \phi} \hat{\phi}, \quad \mathbf{E} = -\frac{\partial A}{\partial t} = +i\omega A \hat{z}.
\]

Either way, the same result

(b) Write an (exact) integral expression for the vector potential (without necessarily doing the integral).

Using the retarded Greens’s function solution (9.2) for the wave equation or the outgoing wave Green’s function solution (9.3) for the Helmholtz equation (these are the same thing),

\[
A = \frac{\mu_0}{4\pi} \int d^3x' dt' \mathbf{J}(x') e^{-i\omega t'} \frac{\delta(t' - t - |x - x'|/c)}{|x - x'|} = \frac{\mu_0}{4\pi} \int d^3x' \mathbf{J}(x) e^{ik|x-x'|} |x - x'|
\]

This is the starting point for any radiation problem. For the wire, \( \rho' = 0 \), and

\[
|x - x'|^2 = \rho^2 + \rho'^2 - 2\rho \rho' \cos(\phi - \phi') + (z - z')^2 = \rho^2 + (z - z')^2
\]

Then,

\[
A = \frac{\mu_0}{4\pi} \int dz' I_0 \hat{z} \frac{e^{ik\sqrt{\rho^2 + (z - z')^2}}}{\sqrt{\rho^2 + (z - z')^2}} = \frac{\mu_0 I_0 \hat{z}}{4\pi} \int dz'' \frac{e^{ik\sqrt{\rho^2 + (z'' - z')^2}}}{\sqrt{\rho^2 + (z'' - z')^2}}.
\]
where from now on every field has an implicit $e^{-i\omega t}$. The last is an adequate “integral expression,” of the form anticipated in (a), but from the hint, with $\xi = kz''$ and $\beta = k\rho$, the integral is [this integral was given on the cover page]

$$
\int_{-\infty}^{\infty} d\xi \frac{e^{i\sqrt{\beta^2 + \xi^2}}}{\sqrt{\beta^2 + \xi^2}} = i\pi H_0^{(1)}(\beta),
$$

and

$$
A = \frac{i\mu_0 I_0 \hat{z}}{4} H_0^{(1)}(k\rho).
$$

(c) What are the electric and magnetic fields near the wire? What distinguishes “near”?

For small argument (3.89, 3.90)

$$
H_0^{(1)}(x) = J_0(x) + iN_0(x) \approx 1 - \frac{x^2}{4} + \cdots + \frac{2i}{\pi} \left[ \ln \frac{x}{2} + \gamma_E - \left( \ln \frac{x}{2} + \gamma_E - 1 \right) \frac{x^2}{4} + \cdots \right]
$$

$$
\approx \frac{2i}{\pi} \ln x + \frac{2i}{\pi} (\gamma_E - \ln 2) + 1,
$$

($\gamma_E = 0.5772156649\ldots$ is Euler’s constant), and to leading order

$$
B = \nabla \times A = \hat{\rho} \frac{\partial}{\partial \rho} \times \left[ \frac{i\mu_0 I_0 \hat{z}}{4} \frac{2i}{\pi} \ln(k\rho) \right] \approx \frac{\mu_0 I_0}{2\pi \rho} \hat{\phi}.
$$

This is the static field of a long, current-carrying wire, modulated by $\cos \omega t = \Re[e^{-i\omega t}]$. At this order the curl vanishes, but if you keep the second order terms in the expansion of $H_0^{(1)}$ or if you just use $E = i\omega A$,

$$
E \approx -\frac{\mu_0 \omega I_0 \hat{z}}{4} \left( \ln \frac{x}{2} + \gamma_E \right).
$$

This vanishes for a static current, $\omega \to 0$. “Near the wire” means closer than a wavelength, $k\rho \lesssim 1$.

(d) What are the electric and magnetic fields far from the wire?

At large argument, from (3.91) the Bessel functions behave as

$$
J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\pi}{4} \right), \quad N_0(x) \approx \sqrt{\frac{2}{\pi x}} \sin \left( x - \frac{\pi}{4} \right),
$$

so that

$$
H_0^{(1)} \approx \sqrt{\frac{2}{\pi x}} e^{i(x - \frac{\pi}{4})} \sqrt{-i} \sqrt{\frac{2}{\pi x}} e^{ix},
$$
and the vector potential is

\[ A = \frac{\mu_0 I_0 \hat{z}}{4} \sqrt{\frac{2i}{\pi k \rho}} e^{ik\rho}. \]

This also follows directly from the large-\( \rho \) limit of the integral in (b), with \( \sqrt{\rho^2 + z''^2} \approx \rho + z''^2/2\rho \), using the Gaussian integral hint (and not worrying about convergence),

\[ \int_{-\infty}^{\infty} d\xi e^{-\xi^2/2\beta^2} = \sqrt{2\pi \beta^2}, \]

\[ A \approx \frac{i\mu_0 I_0 \hat{z}}{4\pi} \int dz'' \frac{e^{ik\rho + ikz''/2\rho}}{\rho} \approx \frac{i\mu_0 I_0 \hat{z}}{4\pi} \frac{e^{ik\rho}}{\rho} \sqrt{\frac{2\pi \rho}{i k}}. \]

Far from the wire, the gradient is dominated by the radial derivative of the phase, \( \nabla \approx ik\hat{\rho} \), and

\[ B = \nabla \times A = ik\hat{\rho} \times A\hat{z} = ik \left( \frac{\sqrt{\mu_0 I_0}}{2\sqrt{2\pi}} \right) \frac{e^{ik\rho}}{\sqrt{k\rho}} (-\hat{\phi}). \]

The electric field is found either from the curl of \( B \) or from the time derivative of the vector potential,

\[ \frac{1}{c^2} \frac{\partial E}{\partial t} = -\frac{i\omega}{c^2} E = \nabla \times B = \nabla \times (\nabla \times A) = k^2 A; \]

either way,

\[ E = i\omega A = i\omega \left( \frac{\sqrt{\mu_0 I_0}}{2\sqrt{2\pi}} \right) \frac{e^{ik\rho}}{\sqrt{k\rho}} \hat{z}. \]

(e) What is the angular distribution of radiation about the wire? What is the rate at which power is radiated per length of wire?

The intensity of power radiated is given by the time-averaged Poynting vector,

\[ \langle S \rangle = \frac{1}{2} \text{Re} \left[ \mathbf{E} \times \mathbf{H}^* \right] = \frac{1}{2} \frac{\omega k k_0^2 |I_0|^2}{8\pi \mu_0 k \rho} \hat{z} \times (-\hat{\phi}) = \frac{\omega}{16\pi \rho} \mu_0 |I_0|^2 \hat{\rho}. \]

The flux of energy is innately real, is radial, and is independent of angle. The power per unit length is

\[ \mathcal{P} = 2\pi \rho \hat{\rho} \cdot S = \frac{1}{8} \mu_0 \omega |I_0|^2. \]

2. A linearly polarized electromagnetic plane wave with frequency $\omega$ and wavenumber $k$ is normally incident from vacuum onto an excellent conductor with permittivity $\epsilon_c = \epsilon_0$, permeability $\mu_c = \mu_0$, and conductivity $\sigma$.

(a) Show that the incident electric and magnetic fields are transverse. How are $\omega$ and $k$ related? How are the electric and magnetic fields $E$ and $B$ related? Compute the time average incident power per unit area in terms of the electric field amplitude $E_0$.

All follows from Maxwell’s equations. Let the wavevector define the $z$-direction; the fields of the plane wave are then $E = E_0 e^{i(kz-\omega t)}$, $B = B_0 e^{i(kz-\omega t)}$. From the divergence equations we have

$$\nabla \cdot E = ik \cdot E = 0, \quad \nabla \cdot B = ik \cdot B = 0.$$  

Thus, the fields must be perpendicular to the direction of $k$, or transverse. From the curl equations we have

$$\nabla \times E = ik \times E = -\frac{\partial B}{\partial t} = i\omega B, \quad \nabla \times B = ik \times B = \frac{1}{c^2} \frac{\partial E}{\partial t} = -\frac{i\omega}{c^2} E.$$  

so

$$k \times E = \omega B, \quad k \times B = -\frac{\omega}{c^2} E.$$  

Combining the curl equations leads to

$$k \times (k \times E) = -k^2 E = k \times (\omega B) = -\frac{\omega^2}{c^2} E,$$

$$k \times (k \times B) = -k^2 B = k \times \left(-\frac{\omega}{c^2} E \right) = -\frac{\omega^2}{c^2} B.$$  

From either of these we conclude

$$k^2 = \frac{\omega^2}{c^2}, \quad k = \frac{\omega}{c}.$$  

With this,

$$\hat{k} \times E = c B, \quad \hat{k} \times B = -\frac{1}{c} E.$$  

The time average power per unit area is found from the Poynting vector (OK, not everything follows directly from Maxwell’s equations),

$$\langle S \rangle = \frac{1}{2} \text{Re}[E \times H^*] = \frac{1}{2\mu_0 \omega} \text{Re}[E \times (k \times E)^*] = \frac{1}{2\mu_0 c} \hat{k}|E|^2 = \frac{\hat{z}|E_0|^2}{2Z_0}.$$  

(b) Part of the incident wave continues into the conducting medium. How are $\omega$ and $k$ related within the conductor? How are $\mathbf{E}$ and $\mathbf{B}$ related within the conductor? What determines whether the material is “excellent” conductor? Compute the time average power as a function of depth within the conducting medium for such a conductor in terms of the electric field amplitude.

Faraday’s law still gives $\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}$. The difference in this case is that within the conductor there is a current obeying Ohm’s Law, $\mathbf{J} = \sigma \mathbf{E}$,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}, \quad \mathbf{k} \times \mathbf{B} = -\left(i \mu_0 \sigma + \frac{\omega}{c^2}\right) \mathbf{E}. $$

Within the conductor we have

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = -k^2 \mathbf{E} = \mathbf{k} \times (\omega \mathbf{B}) = -\left(i \mu_0 \sigma \omega + \frac{\omega^2}{c^2}\right) \mathbf{E},$$

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{B}) = -k^2 \mathbf{B} = -\left(i \mu_0 \sigma + \frac{\omega}{c^2}\right) \mathbf{k} \times \mathbf{E} = -\left(i \mu_0 \sigma \omega + \frac{\omega^2}{c^2}\right) \mathbf{B}. $$

Thus, from both $\mathbf{E}$ and $\mathbf{B}$,

$$k^2 = i \mu_0 \sigma \omega + \frac{\omega^2}{c^2}. $$

For the conductivity term to dominate, we need

$$\sigma \gg \frac{\omega}{c^2 \mu_0} = \varepsilon_0 \omega. $$

This identifies an “excellent” conductor. When this holds, we have

$$k = \pm \sqrt{i \mu_0 \sigma \omega} = \pm \frac{1+i}{\delta}, \quad \delta = \sqrt{\frac{2}{\mu_0 \sigma \omega}}. $$

The electric and magnetic fields are then combinations of $e^{\pm ikz} = e^{\mp z/\delta} e^{\pm iz/\delta}$. For an incident wave from $z < 0$, we select the forward-moving wave, which is also the solution that falls off as $z$ increases,

$$\mathbf{E} = \mathbf{E}_1' e^{iz/\delta} e^{-z/\delta}, $$

where the value of $\mathbf{E}_1'$ is deferred to part (c). The time average power per unit area is

$$\langle S \rangle = \frac{1}{2} \operatorname{Re} [\mathbf{E} \times \mathbf{H}^*] = \frac{\operatorname{Re} [\mathbf{E} \times (\mathbf{k} \times \mathbf{E})^*]}{2 \mu_0 \omega} = \frac{\operatorname{Re} [k \hat{z} |\mathbf{E}|^2]}{2 \mu_0 \omega} = \frac{\hat{z} |\mathbf{E}_1'|^2}{2 \omega \delta Z_0/c} e^{-2z/\delta}. $$
(c) Part of the incident wave is reflected. Write equations relating the fields of the incident, reflected, and transmitted waves. Solve for the reflected field amplitude, to lowest nonvanishing order in $\omega \epsilon_0/\sigma$ or $\omega \delta/c$ (where $\delta$ is the skin depth). What fraction of the incident power is reflected?

Except for ideal conductors, where there can be infinitely thin surface charges and currents, we have

$$\Delta E_\parallel = 0, \quad \Delta D_\perp = 0, \quad \Delta H_\parallel = 0, \quad \Delta B_\perp = 0.$$ 

Let the incident wave be in the $z$-direction, and let the electric fields be in the $z$-direction, with amplitudes $E_0$ incident, $E''_r$ reflected, and $E'_1$ transmitted. The magnetic fields will be in the $y$-direction, with amplitudes

$$B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0, \quad B''_r = -\frac{1}{c} E''_r, \quad B'_1 = \frac{1 + i}{\omega \delta} E'_1.$$ 

Thus, the amplitudes $E'_1$ and $E''_r$ satisfy

$$E_0 + E''_r = E'_1, \quad \frac{1}{c} E_0 - \frac{1}{c} E''_r = E'_1 \frac{1 + i}{\omega \delta}.$$ 

The solution is

$$E''_r = -E_0 + \frac{(1 - i) E_0 \omega \delta/c}{1 + (1 - i) \omega \delta/2c}, \quad E'_1 = \frac{(1 - i) E_0 \omega \delta/c}{1 + (1 - i) \omega \delta/2c}.$$ 

The factor $\omega \delta/c$ is small for a good conductor,

$$\frac{\omega^2 \delta^2}{c^2} = \frac{\omega^2}{c^2} \frac{2 \mu_0 \sigma \omega}{\epsilon_0 \omega} \approx \frac{2 \omega \epsilon_0}{\sigma} \ll 1,$$

and to lowest order the denominators might as well be 1. The reflection coefficient is

$$R = \frac{|E''_r|^2}{|E_0|^2} = \frac{1 - \omega \delta/c + (\omega \delta/c)^2/2}{1 + \omega \delta/c + (\omega \delta/c)^2/2} \approx 1 - \frac{2 \omega \delta}{c}.$$ 

You can also apply the results of Jackson’s Problem 7.6, with effective index of refraction

$$n^2 = \frac{\epsilon_{\text{eff}}}{\epsilon_0} = 1 + \frac{i \sigma}{\epsilon_0 \omega} \approx \frac{i \sigma}{\epsilon_0 \omega}.$$ 

If you don’t know why this is, then you can’t do it this way.
(Bonus): Estimate the reflection coefficient (the ratio of reflected to incident power) numerically for 1 GHz microwaves and for optical frequencies incident on silver, $\sigma = 6.30 \times 10^7 (\Omega m)^{-1}$.

Numerically, for 1 GHz microwaves the reflection coefficient is

$$R_{\text{microwave}} = 1 - \sqrt{\frac{8\varepsilon_0(2\pi f)}{\sigma}} = 1 - \sqrt{\frac{4(4\pi\varepsilon_0)f}{\sigma}} = 1 - \sqrt{\frac{(4)(10^9)}{(9 \times 10^9)(6.3 \times 10^7)}} = 1 - 8.4 \times 10^{-5} = 0.999916.$$

For optical frequencies, take $\lambda = 600$ nm, $f = 5 \times 10^{14}$ Hz, and

$$R_{\text{optical}} = 1 - \sqrt{\frac{(4)(5 \times 10^{14})}{(9 \times 10^9)(6.3 \times 10^7)}} = 1 - 0.053 = 0.947.$$

(d) Let the conductor be a slab of finite thickness $d$, so that there is a transmitted wave on the far side. Write and solve a set of equations relating the modes within the conductor and the transmitted wave on the far side.

Let $E'_+$ and $E'_-$ are the amplitudes of left- and right-moving modes within the conductor at the exit. This time, boundary conditions on the electric and magnetic fields give

$$E'_+ + E'_- = E''_t, \quad \frac{1+i}{\omega \delta} E'_+ - \frac{1+i}{\omega \delta} E'_- = \frac{1}{c} E''_t.$$

These give

$$\frac{E'_+}{E''_t} = \frac{1}{2} + \frac{(1 - i)\omega \delta}{4c}, \quad \frac{E'_-}{E''_t} = \frac{1}{2} - \frac{(1 - i)\omega \delta}{4c}.$$

We learned in (c) that $\omega \delta / c$ is small for a good conductor, and so at exit the two modes within the conductor have very close to equal amplitudes.
(e) If the two modes within the conductor are of comparable amplitude at the exit side, one is exponentially smaller at the entry side. If this term is negligible, so that the field in the conductor is the same as in (b)-(c) for an infinite slab, find the transmission coefficient (the ratio of transmitted to incident power).

At the entry to the conductor, the right-moving mode is larger by the exponential factor \( \exp(1 + i)d/\delta \), and the left-moving mode is smaller by the same factor. If we ignore the left-moving mode at the first interface, then the right-moving amplitude \( E'_1 \) is as before, and so with \( E'_1 = e^{(1+i)d/\delta} E'_+ \), \( E''_t \) is related to \( E_0 \) as

\[
E'_t = \frac{2 e^{-(1+i)d/\delta} E'_1}{1 + (1 - i)\omega\delta/2c} = \frac{2 e^{-(1+i)d/\delta} (1 - i) \omega\delta/c E_0}{[1 + (1 - i)\omega\delta/2c]^2}.
\]

The transmission coefficient is then

\[
T = \frac{|E''_t|^2}{|E_0|^2} = \frac{8 (\omega\delta/c)^2 e^{-2d/\delta}}{(1 + \omega\delta/2c)^2 + (\omega\delta/2c)^2)^2} \approx \frac{8 (\omega\delta/c)^2 e^{-2d/\delta}}{1 + 2\omega\delta/c}.
\]

Jackson’s Problem 7.5 at the end of Chapter 7 considers this situation, with attention to the exponentially small factor \( e^{-d/\delta} \), while ignoring factors of \( \omega\delta/c \).