Due at the start of class on Friday, January 20. No credit will be available for homework submitted after the start of class on Wednesday, January 25.

Answer all four questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

This assignment is primarily designed to provide practice with standard mathematical techniques encountered in wave mechanics. You may find useful the following integrals:

\[
\int_0^\infty x^{2n} \exp(-x^2/a^2) \, dx = \sqrt{\pi} \frac{(2n)!}{n!} \left(\frac{a}{2}\right)^{2n+1}, \quad \int_0^\infty \frac{dx}{(a^2 + x^2)^{n+1}} = \frac{\pi (2n - 1)!!}{2^n 1^{2n+1} n!}
\]

where \(a\) is real, \(n\) is a non-negative integer, \(n! = n \cdot (n - 1)!\) with \(0! = 1\), and \(n!! = n \cdot (n - 2)!!\) with \(0!! = (-1)!! = 1\).

1. Consider the wave function

\[
\Psi(x, t) = A \exp[-(x^2/2a^2 + iEt/\hbar)],
\]

where \(a\) is a real length scale and \(E\) is a real energy.

(a) Find the positive, real constant \(A\) that normalizes the wave function.

(b) Calculate \(\langle x \rangle\), \(\langle x^2 \rangle\), and \(\sigma_x\).

(c) Calculate \(\langle p \rangle\), \(\langle p^2 \rangle\), and \(\sigma_p\).

(d) Verify that \(\sigma_x\) and \(\sigma_p\) satisfy the uncertainty principle.

2. Redo Question 1 for

\[
\Psi(x, t) = \frac{A}{x^2 + a^2} \exp(-iEt/\hbar).
\]

3. In order to better characterize a probability distribution \(P(x)\), it is common to calculate central moments

\[\mu_n = \langle (x - \langle x \rangle)^n \rangle\]

beyond the variance \(\mu_2 = \sigma^2\). The skewness \(\mu_3/\sigma^3\) measures the asymmetry of the distribution about the mean, and the kurtosis \(\mu_4/\sigma^4\) measures how “fat” or “skinny” the tails of the distribution are.

For parts (a) and (b) below, provide a reasoned argument rather than a detailed calculation.

(a) Show that for the wave function in Eq. (1), both the position distribution \(P(x)\) and the momentum distribution \(P(p)\) have zero skewness and finite (but nonzero) kurtosis.

(b) Show that for the wave function in Eq. (2), \(P(p)\) has finite kurtosis but \(P(x)\) has infinite kurtosis. (Infinite kurtosis of the spatial probability distribution indicates that there is a significant chance of measurement of \(x\) yielding a result very far from the mean.)
4. Suppose that in some region \( x_1 < x < x_2 \), a point-like particle of mass \( m \) is described by the wave function

\[
\Psi(x, t) = A \sin(\pi x/a) e^{-i\omega t} + B \sin(2\pi x/a) e^{-4i\omega t},
\]

where \( a \) and \( \omega \) are positive reals, and \( A \) and \( B \) are complex. [Note that if this form of \( \Psi(x, t) \) were to extend all the way to \( x = \pm \infty \), the wave function would not be normalizable. We will assume that \( \Psi(x, t) \to 0 \) for \( x \ll x_1 \) and for \( x \gg x_2 \).]

(a) Calculate the probability density \( \rho(x, t) \) and the probability current \( j(x, t) \) within the region \( x_1 < x < x_2 \). Cast your answers in forms that make clear that both quantities are real.

(b) If one of \( A \) or \( B \) is zero, then the wave function describes a stationary state. Show that in this case, \( \rho(x, t) = \rho(x) \) independent of time, and \( j(x, t) = 0 \).