The Hamiltonian Formulation of Classical Mechanics: Key Ideas

1. The state of a system containing \( n \) degrees of freedom (e.g., \( N \) unconstrained particles in \( d \) spatial dimensions correspond to \( n = Nd \)) is described by \( 2n \) canonical variables:
   - \( n \) generalized coordinates, \( q = (q_1, q_2, \ldots, q_n) \).
   - \( n \) conjugate momenta, \( p = (p_1, p_2, \ldots, p_n) \).

   N.B. The momentum conjugate to \( q_i \) is formally defined in the Lagrangian picture: \( p_i = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_i} \).

2. The dynamics of the system are governed by \( 2n \) first-order partial differential equations, Hamilton’s equations:
   \[
   \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}.
   \]
   These equations replace the \( n \) second-order PDE’s corresponding to Newton’s second law, \( F = ma \), applied to each particle.

3. In many (but not all) situations, the Hamiltonian \( H(q, p, t) \) is the total energy of the system.

4. The equation of motion of any dynamic variable \( \omega(q, p, t) \) is
   \[
   \frac{d\omega}{dt} = \{\omega, H\} + \frac{\partial \omega}{\partial t},
   \]
   where the Poisson bracket is defined to be
   \[
   \{\omega, \lambda\} = \sum_i \left( \frac{\partial \omega}{\partial q_i} \frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \frac{\partial \lambda}{\partial q_i} \right).
   \]
   Hamiltonian’s equations are a special case of Eq. (1):
   \[
   \frac{dq_i}{dt} = \{q_i, H\}, \quad \frac{dp_i}{dt} = \{p_i, H\}.
   \]
   Note also that
   \[
   \{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}.
   \]

5. A dynamic variable \( \omega \) that has no explicit time dependence (\( \partial \omega/\partial t = 0 \)) and has a vanishing Poisson bracket with the Hamiltonian is a conserved quantity. The existence of conserved dynamic variables is intimately tied to symmetries of the Hamiltonian. Specifically, \( \omega \) is conserved if \( H \) is invariant under the infinitesimal transformation
   \[
   q_i \rightarrow q_i + \varepsilon \frac{\partial \omega}{\partial p_i} \equiv q_i + \varepsilon \{q_i, \omega\}, \quad p_i \rightarrow p_i - \varepsilon \frac{\partial \omega}{\partial q_i} \equiv p_i + \varepsilon \{p_i, \omega\}.
   \]