Due by 5 p.m. on Friday, September 20. Partial credit will be available for solutions submitted by 5 p.m. on Monday, September 23.

Answer all questions. To gain maximum credit you should explain your reasoning and show all working. Please write neatly and include your name on the front page of your answers.

1. Projection operators play an important role in quantum mechanics. For instance, they enter the completeness relation: $I = \sum_j |j\rangle\langle j|$, where $\{|j\rangle\}$ is a complete basis; and the spectral representation of any analytic function $f$ of a normal operator $\Omega$: 

$f(\Omega) = \sum_j f(\omega_j)|\omega_j\rangle\langle\omega_j|$, where $\{\omega_j\}$ is the complete basis of eigenkets of $\Omega$.

Prove the following properties of any projection operator $P_\phi = |\phi\rangle\langle\phi|$ acting on an inner product space that has a basis:

(a) $P_\phi$ is idempotent: $P_\phi^2 = P_\phi$.

(b) $P_\phi$ is positive semi-definite: $\langle\psi|P_\phi|\psi\rangle \geq 0$ for all $|\psi\rangle$ in the vector space.

(c) $P_\phi$ has no inverse (unless the vector space is one-dimensional).

2. Consider a Hermitian operator $\Omega$ having a set of discrete eigenvalues $\{\omega_j\}$.

(a) Show that $\prod_j (\Omega - \omega_j I) = 0$ (the null operator), irrespective of whether any of the eigenvalues are degenerate. Show that the product over all eigenvalues above can be replaced by a product over all distinct eigenvalues $\omega$:

\[ \prod_\omega (\Omega - \omega I) = 0. \]

(b) Show that $\Omega$ satisfies its own characteristic equation.

(c) Show that

\[ \prod_{\omega' \neq \omega} \frac{\Omega - \omega'I}{\omega - \omega'} \equiv P_\omega, \]

where $P_\omega$ is the projection operator onto the subspace of eigenkets having eigenvalue $\omega$.

(d) Finally, show that in a vector space of finite dimension $n$, any function $f(\Omega)$ can be represented as a polynomial in $\Omega$ of degree less than $m$, where $m$ is the number of distinct eigenvalues of $\Omega$ ($m \leq n$).

Hint: Parts (a)–(d) will be greatly simplified by a judicious choice of basis in which to work (even though the results are basis-independent).

3. Shankar Ex. 4.2.1.

As Shankar says, this is a very important problem. It provides a concrete, but mathematically simple, illustration of many aspects of the postulates of quantum mechanics. Therefore, it will be graded even though the answers are given in the book.

4. Shankar Exs. 4.2.2 and 4.2.3.